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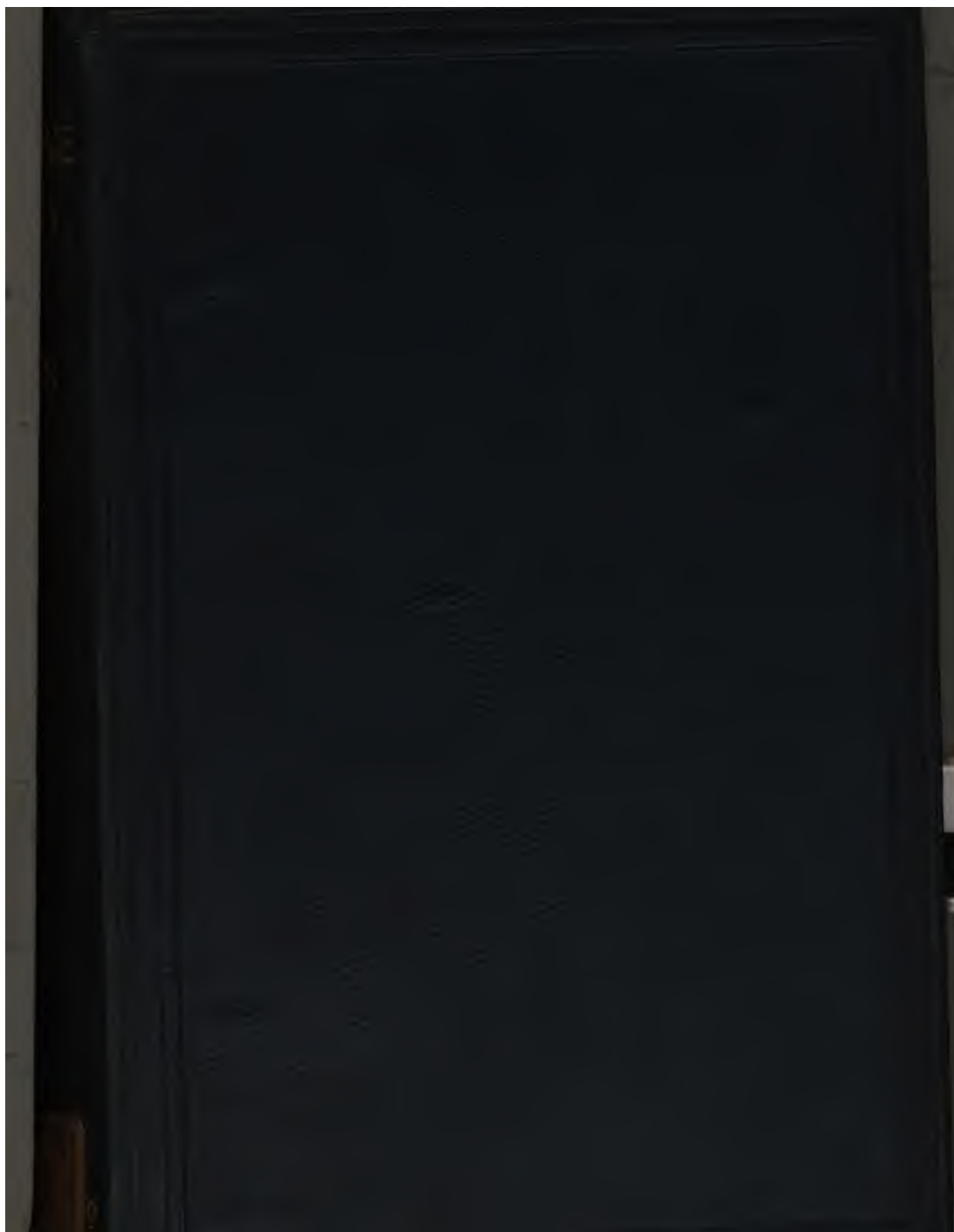
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**CONSTRUCTIVE GEOMETRY**  
**OF**  
**PLANE CURVES.**



# CONSTRUCTIVE GEOMETRY

OF

## PLANE CURVES

WITH NUMEROUS EXAMPLES

BY

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## P R E F A C E.

THE appearance of another text-book on Geometry may perhaps be considered to demand an apology, but I venture to hope that an examination of the following pages will shew them to differ considerably from any existing treatise. The extending use of graphic methods in the solution of many practical engineering problems has appeared to me to demand a corresponding extension in the practice of drawing the curves on which such solutions may frequently depend, and, though the properties of conic sections have been discussed thoroughly both geometrically and analytically, there is so far as I am aware no book treating of the actual delineation of the curves from given data to anything like the extent here attempted. Independently however of their applied use, the problems generally will, I think, be found useful merely as drawing exercises in science and other schools. A great deal of attention is devoted to the construction of regular polygons, circles packed into another circle and similar fancy figures, by methods which no practical draughtsman ever uses, while the construction of an ellipse is at the most limited to drawing it from the principal axes or from a pair of conjugate diameters; and the time spent on these and similar exercises might, I think,

be more profitably devoted to work bringing out the nature and properties of this and other curves.

I can say from experience that the practice of sketching a curve freehand through a series of previously found points is a most valuable element in teaching mechanical drawing, while the finding the points furnishes abundant exercise in handling square and compasses, and impresses on the student in a very striking manner the necessity for neatness and accuracy in their use.

Each problem may of course be drawn on paper without reference to the proof of the principle on which its construction depends, but I consider that for the advanced student at any rate it must be much more satisfactory to work with as complete an insight as possible into the methods he is using instead of groping along by mere rule of thumb, so that in nearly all cases notes in proof of the property made use of have been added, although such proofs may be found in numerous published works, and are indeed so completely common property that I have not thought it necessary to give direct references to the pages from which they have been taken.

I cannot however here omit to notice my indebtedness to Dr Salmon's classical work on Conic Sections, or to Chasles' *Géométrie Supérieure* for the chapter on Anharmonic Ratio and the Anharmonic Properties of Conics. Chap. VIII. will, I hope, convince a draughtsman that he can if he likes make use of an engine very little known in England and of enormous power. The methods of Modern Geometry deserve to be brought into much closer relation with the drawing-board than has hitherto been the case.

The chapter on Plane Sections of the Cone and Cylinder involves some elementary notions of Solid Geometry or Orthographic Projection, but the explanations given will, I hope, enable the average student to work through the chapter

without referring to any special treatise on Projection. The ordinary pseudo-perspective diagrams usually given in books on Conics are I think unsatisfactory, and the method of referring the solid to two rectangular planes seems to me in every way preferable. When the mental conception of a plan and elevation is once thoroughly realised the student is well repaid by the exactness with which he is able to lay down on paper any point or line on the surface of the cone.

The later chapters cannot be read without some knowledge of trigonometry, but the practice of translating a trigonometrical expression into something which can be represented to the eye is a valuable one, and the hints given in the chapter on the Graphic Solution of Equations will I trust be found useful.

My warmest thanks are due to my friend and colleague Professor Minchin for much valuable advice and assistance most freely and readily given: without his help the book would have been much less complete than it is, whatever its imperfections may be found to be.

It would be too much to hope that a work of this character should have been compiled and gone through the press without some errors creeping in. I hope they are not more numerous than from the nature of the case may be considered unavoidable, and I shall be thankful for any such being brought to my notice.

COOPERS HILL,  
Oct. 1885.





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# CONSTRUCTIVE TREATISE ON PLANE CURVES.

## CHAPTER I.

### INTRODUCTORY.

THE Instruments required for the accurate representation on paper of almost all known curves are few in number and of simple construction. For accurate work however it is essential they should be of good quality, and be kept in good order. A limited number of good instruments is in every way to be preferred to a larger number of inferior articles, and where economy is an object therefore, in preference to the usual large and small single jointed compasses found in cheap boxes of mathematical instruments the author strongly recommends the purchase of one medium size, double jointed pair of compasses with pen and pencil points, which can be used for both large and small circles if care be taken to adjust the legs so that the lower portions of both may be perpendicular to the paper. This is a *sine qua non* for good work and it is of course impossible with the ordinary single jointed instruments. In addition to the above a pair of dividers, a drawing pen for inking in straight lines, a protractor which should also contain a diagonal scale of half-inches, a couple of set squares ( $45^\circ$  and  $60^\circ$ ), pencil and paper may be considered a complete equipment for the work of the following pages.

More may be learnt as to the proper way of handling these tools by ten minutes' observation of a practised draughtsman than from pages of explanation, but failing the opportunity of this practical instruction, the following hints may be useful.

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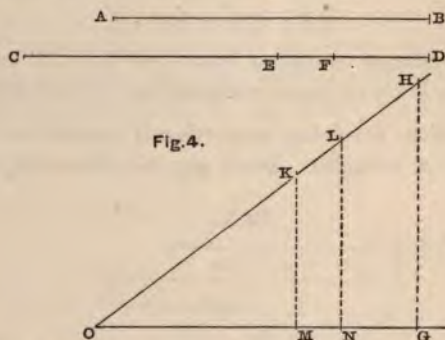
A similar construction will obviously give a third proportional to two given lines  $AB$ ,  $CD$ ; i.e. a line of length ( $l$ ) such that

$$AB : CD :: CD : l,$$

or that the rectangle contained by  $AB$  and  $l$  shall be equal in area to the square on  $CD$ ; the only difference being that in this case the lengths  $OH$  and  $OK$  will be equal to each other.

PROBLEM 3. (Fig. 4.) *To divide a line of given length ( $AB$ ) similarly to a given line  $CD$  divided in any manner as at  $E, F, \dots$  (There may be any manner of points of division.)*

Draw any two lines as  $OG$  and  $OH$ . Make  $OG = AB$ ,  $OH = CD$ ,  $OK = CE$ ,  $OL = CF \dots$  and draw  $KM$ ,  $LN \dots$  parallel to  $HG$ . The



line  $OG$ , i.e.  $AB$  will be divided in  $M, N \dots$  similarly to  $CD$  in  $E, F \dots$

PROBLEM 4. (Fig. 1.) *To draw a line through a given point and through the intersection of two given lines.*

It is of course the simplest possible thing to do this when the actual point of intersection of the two lines is available. As in Problem 1 however it is frequently necessary to draw a line the direction of which depends on an inaccessible point. Let  $AB, CD$  be the two given lines and  $M$  the given point. ( $M$  may be between the lines as in fig. 1 or on the farther side of either with regard to the other.) Draw any line through  $M$  meeting the given lines in  $N$  and  $O$ , and at any convenient distance from  $M$

draw a second line parallel to  $NO$  meeting the given lines as at  $A$  and  $C$ . If we divide  $AC$ , as in  $Q$ , in similar segments to those in which  $M$  divides  $NO$  (Problem 3) the line  $QM$  will be the required line passing through the intersection of  $AB$  and  $CD$ . The most convenient method for dividing  $AC$  is probably thus : join  $CN$ , draw  $PM$  through  $M$  parallel to  $CD$  and meeting  $CN$  in  $P$ , and through  $P$  draw  $PQ$  parallel to  $AB$  meeting  $AC$  in  $Q$ .  $AC$  is obviously divided in  $Q$  similarly to  $NC$  in  $P$  and therefore to  $NO$  in  $M$ .

**PROBLEM 5.** (Fig. 5.) *To find the geometric mean between two given lines  $AB$ ,  $CD$ , i.e. to find a line of length ( $l$ ) such that*

$$AB : l :: l : CD,$$

or that the square on  $l$  shall be equal in area to the rectangle contained by  $AB$  and  $CD$ .

Draw any straight line  $EOF$  and set off on it on opposite sides from  $O$ ,  $OE = AB$ ,  $OF = CD$ . On  $EF$  describe a semicircle

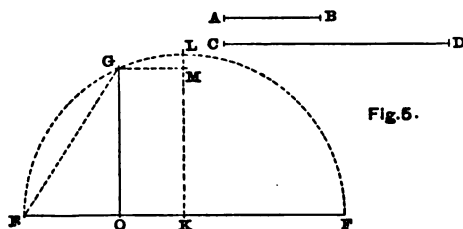


**Fig.6.**

and from  $O$  draw  $OG$  perpendicular to  $EF$  meeting the circumference in  $G$ .  $OG$  will be the required mean proportional or geometric mean. For, since the angle in a semicircle is a right-angle (Euclid III. 31),  $\therefore$  the angles  $OEG, EGO$  are together equal to the angles  $EGO, OGF$ , and  $\therefore$  the angle  $OEG =$  the angle  $OGF$ ,  $\therefore$  the right-angled triangles  $OEG, OGF$  are similar and

$$\therefore EO : OG :: OG : OF,$$

i.e.

$$AB : OG :: OG : CD.$$

PROBLEM 6. *To divide a given line so that the rectangle contained by its segments is equal to the square on a given line which*

must obviously be not greater than half the line to be divided (fig. 5).

This is the converse of the last problem. Let  $EF$  be the given line, on it describe a semicircle. Draw the radius  $KL$  perpendicular to  $EF$  and on it make  $KM$  equal to the side of the required square. Through  $M$  draw a parallel to  $EF$  meeting the circle in  $G$  and from  $G$  drop a perpendicular on  $EF$  meeting it in  $O$ ,  $O$  will be the required point of division.

The construction is obvious from the last problem.

**PROBLEM 7.** (Fig. 6.) *To divide a line medially, or in extreme and mean proportion, i.e. to find a point ( $F$ ) in a line  $AB$  such that*

*the whole line  $AB$  : the greater segment ( $BF$ )*

*::  $BF$  : the lesser segment ( $AF$ ),*

*or that the rectangle contained by the whole line and the lesser segment is equal in area to the square on the greater segment.*

Bisect  $AB$  in  $C$ , from  $A$  draw  $AD$  perpendicular to  $AB$  and make  $AD = AC = \frac{1}{2}AB$ . Join  $BD$  and on it from  $D$  cut off

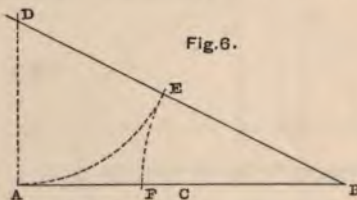


Fig. 6.

$DE = DA$ ; from  $B$  on  $BA$  cut off  $BF = BE$ .  $F$  will be the required point. This construction is simplified from Euclid II. 11, the proof may be shewn thus.

The sq. on  $BD$  = sq. on  $AB$  + sq. on  $AD$  (Euclid I. 47).

Also " = sq. on  $EB$  + sq. on  $ED$  + 2 rect.  $EB \cdot ED$ ,  
but  $ED = AD$  and  $EB = FB$ , (Euclid II. 4),

$\therefore$  sq. on  $AB$  = sq. on  $FB$  + 2 rect.  $AF \cdot AD$ .

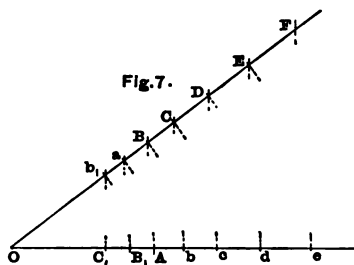
Again sq. on  $AB$  = sq. on  $FB$  + sq. on  $AF$  + 2 rect.  $AF \cdot FB$   
(Euclid II. 4),

= sq. on  $FB$  + rect.  $AF (AF + FB)$   
+ rect.  $AF \cdot FB$ .

$\therefore 2 \text{ rect. } FB \cdot AD = \text{rect. } AF \cdot AB + \text{rect. } AF \cdot FB,$   
 i. e.  $\text{rect. } FB \{2AD - AF\} = \text{rect. } AF \cdot AB,$   
 but  $2AD = AB$  and  $AB - AF = FB,$   
 $\therefore$  finally sq. on  $FB = \text{rect. } AF \cdot AB.$

PROBLEM 8. (Fig. 7.) *To find graphically a series of terms in geometrical progression, being given either two successive terms or one term and the common ratio.*

Draw two lines  $Oe, Of$  meeting in  $O$  at any convenient angle. On one mark off the 1st given term as  $OA$ , and on the other the



2nd given term as  $OB$ , or if the common ratio be given a length  $OB = 1\text{st term multiplied by the common ratio}.$

[In the figure  $OA$  the first term  $= 2$ , and  $OB = 2.4$ ; the common ratio therefore is  $1.2$ , the unit being  $3''/8$ .]

With centre  $O$  and radius  $OB$  describe an arc cutting  $OA$  in  $b$ ; through  $b$  draw  $bC$  parallel to  $AB$  cutting  $OB$  in  $C$ .  $OC$  will be the required third term of the series. Similarly make  $Oc$  on  $OA = OC$  and through  $c$  draw  $cD$  parallel to  $AB$  cutting  $OB$  in  $D$ ,  $OD$  will be the required fourth term, and so on in succession. Terms on the other side of  $OA$  can also be determined as shown at  $OB_1, OC_1$ , &c.

The construction evidently depends on the similarity of the triangles  $OAB, ObC$ , &c.

by which  $OC : OB :: Ob : OA,$

i. e. since  $Ob = OB, OB^2 = OA \cdot OC,$



similarly 
$$\frac{BF}{FC} = \frac{\sin BOF}{\sin COF} = \frac{\sin BOF}{\sin AOD},$$

and 
$$\frac{AE}{EC} = \frac{\sin AOE}{\sin EOC} = \frac{\sin BOF}{\sin BOD} = \frac{AD}{BD} \cdot \frac{BF}{FC}.$$

It follows of course that in any triangle if lines be drawn from the vertices  $ABC$  meeting the opposite sides in  $F, E, D$  and all passing through the same point  $O$ ,  $AD \cdot BF \cdot CE = DB \cdot FC \cdot EA$ , i.e. that the continued products of the alternate segments taken in order are equal.

**PROBLEM 10.** *To determine graphically the square root of any number ( $n$ ), i.e. to determine a line the length of which : length of a line containing  $n$  units measured on any scale :: 1 :  $\sqrt{n}$ .*

This is sometimes, though misleadingly, called determining the square root of a given line. The fact is that the expression the square root of a given line has no meaning unless we take the line to represent, by the number of units it contains, a given area; and then the line to be found is the side of a square, the number of square units in which is equal to the number of units contained in the line—the same scale of course being used for each. If a triangle  $ABC$  be drawn, right angled at  $A$  and having the sides  $AB, AC$  each one inch long, the side  $BC$  is the side of a square of two square inches area, and in this sense  $BC$  may be said to be the square root of a line two inches long, or of the number 2, the unit being one inch, but if the unit be half-an-inch the same line  $BC$  represents the square root of 8, since (Euc. I. 46)  $BC^2 = AB^2 + AC^2 = 2^2 + 2^2 = 8$ .

If we use a diagonal scale of half-inches, the length  $BC$  may be read on it to two places of decimals, and the number so obtained is the square root of 8 to two decimal places. Any question relating to the square root of a number, must therefore always be taken as involving the application of some particular scale. The square root of any proposed number can be found by splitting the number up so as to make it equal to the sum or difference of two or more squares, and then constructing right-angled triangles having sides equal to the sides of these

squares. Thus  $\sqrt{3} = \sqrt{4-1} = \sqrt{2^2-1^2}$  so that if a right-angled triangle  $ABC$  be drawn, right angled at  $C$  and having  $AB = 2$  inches, and  $AC = 1$  inch,  $BC$  represents  $\sqrt{3}$ , an inch being the unit. (The triangle may be constructed by drawing a semicircle on  $AB$  as diameter and making  $AC$  in it = 1 inch.) If the unit is half-an-inch  $BC$  represents  $\sqrt{AB^2-AC^2}$ , i.e.  $\sqrt{4^2-2^2}$  or  $\sqrt{12}$ .

$\sqrt{5} = \sqrt{3^2-2^2}$ , i.e. is the perpendicular of a right-angled triangle the hypotenuse of which is 3 and the base of which is 2, or it may be determined as the hypotenuse of a right-angled triangle one side of which is 2 and the other 1, since  $\sqrt{5} = \sqrt{2^2+1^2}$ . If we halve the unit the same line would represent  $\sqrt{20}$ .

$\sqrt{6} = \sqrt{2^2+\sqrt{2}^2}$ , i.e. if  $\sqrt{2}$  be first determined,  $\sqrt{6}$  is the hypotenuse of a right-angled triangle the sides of which are 2 and  $\sqrt{2}$ , or it may be determined from  $\sqrt{6} = \sqrt{3^2-\sqrt{3}^2}$ .

$\sqrt{7} = \sqrt{2^2+\sqrt{3}^2}$ , and can be determined if  $\sqrt{3}$  is known.

$\sqrt{8}$  has already been given.

$\sqrt{10} = \sqrt{3^2+1^2}$ .

$\sqrt{11} = \sqrt{4^2-\sqrt{5}^2}$ , and can be determined if  $\sqrt{5}$  is known.

$\sqrt{12}$  has been given above; and the method is probably sufficiently exemplified by the above, but we will take two examples of larger numbers

$\sqrt{47} = \sqrt{6^2+\sqrt{11}^2}$ , thus being made to depend on  $\sqrt{11}$ .

$\sqrt{179} = \sqrt{13^2+\sqrt{10}^2}$  thus being made to depend on  $\sqrt{10}$ , it might also be written  $= \sqrt{11^2+\sqrt{47}^2}$  or could be determined in other ways. No definite instructions can be given as to the best mode of working in any particular case, but as a rule triangles having sides of nearly equal magnitude should be selected, since the intersections of lines cutting at very acute angles cannot be accurately determined.



**DEFINITION.** Three magnitudes are said to be in harmonic progression when the first is to the third as the difference between the first and second is to the difference between the second and third : and the second magnitude is said to be an harmonic mean between the first and third.

Thus if the magnitudes represented by the lengths of three lines (as  $AB$ ,  $AC$ ,  $AD$ , fig. 9) are in harmonic progression and the lines be superimposed with a common extremity as in that fig. :— then

$$AB : AD :: BC : CD.$$

The reciprocals of magnitudes in harmonic progression are in arithmetic progression and conversely :—for, if  $AB$ ,  $AC$ ,  $AD$  are in harmonic progression then by definition

$$AB : AD :: BC : CD, \text{ or } \frac{BC}{AB} = \frac{CD}{AD},$$

and if  $\frac{1}{AB}$ ,  $\frac{1}{AC}$ ,  $\frac{1}{AD}$  are in arithmetic progression then by

definition,

$$\frac{1}{AB} + \frac{1}{AD} = \frac{2}{AC},$$

but this may be written

$$\frac{1}{AB} - \frac{1}{AC} = \frac{1}{AC} - \frac{1}{AD},$$

or

$$\frac{AC - AB}{AB \cdot AC} = \frac{AD - AC}{AC \cdot AD},$$

or

$$\frac{BC}{AB} = \frac{CD}{AD},$$

an identical expression with the above.

**PROBLEM 11.** (Fig. 9.) *To find the harmonic mean between two given lines  $AB$ ,  $AD$ , i.e. to find a line of length  $l$  such that*

$AB : AD :: \text{the difference between } AB \text{ and } l$

$: \text{the difference between } AD \text{ and } l.$

Set off the given lengths from the same point ( $A$ ) on any line and in the same direction along it, as  $AB$ ,  $AD$ . Take any point  $E$  outside  $AD$  and join  $AE$ ,  $DE$ . Through  $B$  draw  $FBG$  parallel





*Transversal*, and every transversal of a harmonic pencil is divided harmonically in the points in which it intersects the lines of the pencil : i. e. the four points of intersection constitute a *Harmonic Range*. For in fig. 9 draw any transversal as  $HKLM$ , and through  $K$  draw  $fKg$  parallel to  $ED$  and therefore to  $FG$ , meeting  $EA$ ,  $EC$  in  $f$  and  $g$ . Obviously since  $BF = BG$ ,  $\therefore Kf = Kg$ .

By similar triangles  $HKf$ ,  $HME$

$$HK : HM :: fK : EM,$$

and by similar triangles  $KLg$ ,  $MLE$

$$KL : LM :: gK : EM,$$

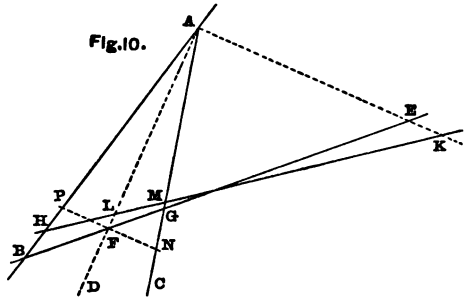
but

$$Kf = Kg,$$

$$\therefore HK : HM :: KL : LM,$$

or  $HKLM$  constitute a Harmonic Range.

A particular case of a Harmonic Pencil is furnished by the pencil formed of two straight lines and the bisectors of the angles between them, as shewn in fig. 10, where  $AD$  bisects the angle



$BAC$  and  $AE$  is drawn perpendicular to  $AD$ , and therefore bisecting the exterior angle between  $AC$  and  $BA$  produced. For draw any transversal as  $BFGE$ , and through  $F$  draw  $PFN$  parallel to  $AE$  and meeting  $AB$ ,  $AC$  in  $P$  and  $N$ .

Then  $PF = FN$  and

$$BF : BE :: PF : AE, \text{ by similar triangles } BPF, BAE,$$

$$FG : GE :: FN : AE, \text{ by similar triangles } FGN, EGA,$$

$$\therefore BF : BE :: FG : GE,$$

or the pencil is harmonic.

A line of given length may obviously be divided harmonically in an infinite number of ways, since a line of length  $HK = BE$  can be drawn from any point  $H$  on  $AB$  to terminate on  $AE$  and

$$HL : HK :: LM : MK.$$

*Harmonic Properties of a complete Quadrilateral.*

If  $FBeA$ ,  $FD_eC$  be harmonic ranges (fig. 11), the straight lines  $AC$ ,  $ee_1$ ,  $BD$  meet in a point, as also  $AD$ ,  $BC$  and  $ee_1$ .

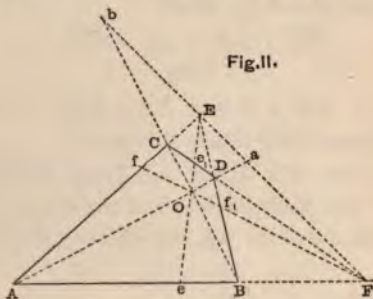


Fig. 11.

For if  $BD$ ,  $AC$  meet in  $E$ , draw  $Ee$ ; then the pencil  $E(AeBF)$  is harmonic and  $FC$  is a transversal, so that  $e_1$  must lie on  $Ee$ .

Similarly if  $AD$  and  $BC$  meet in  $O$ , the pencil  $O(AeBF)$  is harmonic and  $FC$  a transversal, so that  $e_1$  must lie on  $Oe$ .

If  $ABCD$  is any quadrilateral,  $E$  the intersection of the sides  $AC$  and  $BD$ ,  $F$  of the sides  $AB$  and  $CD$ ,  $O$  the intersection of the diagonals  $AD$  and  $BC$ ; it follows conversely that  $EA$ ,  $EO$ ,  $EB$ ,  $EF$  form a harmonic pencil, as also  $FE$ ,  $FC$ ,  $FO$  and  $FA$ . If  $EO$  meet  $AB$  in  $e$  and  $CD$  in  $e_1$ ,  $AeBF$  and  $Ce_1DF$  are therefore harmonic ranges, and if  $FO$  meet  $AC$  in  $f$  and  $BD$  in  $f_1$ ,  $AfCE$  and  $Bf_1DE$  are both harmonic ranges.

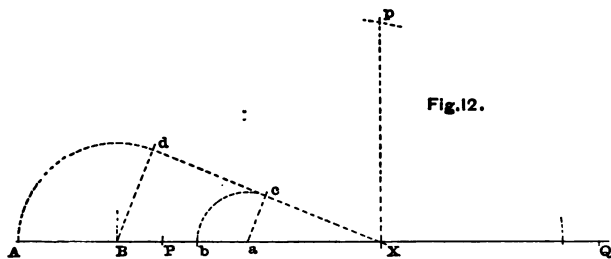
Further if  $AD$  meet  $FE$  in  $a$  and  $BC$  meet it in  $b$ ,  $BOCb$  is a harmonic range since it is a transversal of the pencil  $F(ECfA)$ , therefore  $AF$ ,  $Aa$ ,  $AE$  and  $Ab$  form a harmonic pencil, and therefore  $FaEb$  is a harmonic range, i. e.  $FE$  is divided harmonically in  $a$

DEF. A system of pairs of points  $Aa, Bb$ , &c. on a straight line such that  $XA \cdot Xa = XB \cdot Xb = \dots = XP^2 = XQ^2$  is called a system in Involution, the point  $X$  being called the *centre*,  $P$  and  $Q$  the *foci* of the system, and any two corresponding points  $A, a$ , *conjugate points*.

PROBLEM 13. *Two pairs of conjugate points  $A, a$  and  $B, b$ , being given, to find the centre and foci of the involution.*

The existence of a focus is only possible when both points of a pair are on the same side of the centre, and hence two cases arise, 1st, in which one pair of points lies within the other, and 2nd in which each pair lies wholly outside the other.

Case 1. (Fig. 12.) Let  $ab$  be less than  $AB$ . Through  $a$  the extreme point of the range draw any line  $ac$ , and through  $B$  the



more distant from  $a$  of the second pair of points draw a parallel line  $Bd$ . Make  $ac = ab$ ,  $Bd = BA$ , then  $dc$  will intersect  $Abba$  in  $X$  the required centre—for

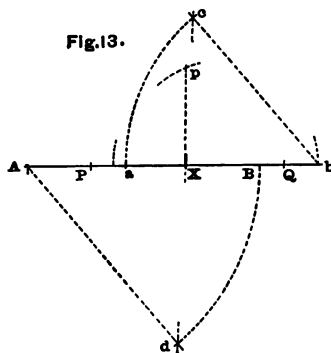
$$\begin{aligned} Xa : ac &:: XB : Bd, \\ \therefore Xa : ab &:: XB : BA, \\ \therefore Xa + ab : Xa &:: XB + BA : XB, \\ \text{i.e. } Xb : Xa &:: XA : XB, \end{aligned}$$

therefore by definition  $X$  is the centre of the system.

Take a mean proportional between either  $XA$  and  $Xa$  or  $XB$  and  $Xb$ , which determines the distance  $XP$  and  $XQ$  from  $X$  of the foci.

18 CENTRE AND FOCI OF POINTS IN INVOLUTION.

*Case 2.* (Fig. 13.) Through the extreme points of the system draw any two parallel lines as  $bc$ ,  $Ad$ . Make  $bc = ba$  the distance



from  $b$  of the nearer point of the opposite pair and make  $Ad = AB$  the distance from  $A$  of the similar point, then  $cd$  will cut  $Ab$  in  $X$  the required centre—for

$$XA : Ad :: Xb : bc,$$

$$\text{i.e. } XA : AB :: Xb : ab,$$

$$\therefore XA : AB - XA :: Xb : ab - Xb,$$

$$\text{or} \quad XA : XB :: Xb : Xa.$$

The foci must be determined as in Case 1.

$$\text{Since} \quad XA : XP :: XP : Xa,$$

$$\therefore XA - XP : XA + XP :: XP - Xa : XP + Xa,$$

$$\text{i.e. } AP : AQ :: Pa : aQ,$$

or each pair of conjugate points forms, with the foci of the system, a harmonic range.

It follows of course that if  $APaQ$  be an harmonic range and  $X$  the centre point of  $PQ$ ,

$$XA \cdot Xa = XP^2 = XQ^2.$$

The following relations between two pairs of conjugate points  $Aa$  and  $Bb$ , and their centre  $X$  and foci  $P$  and  $Q$  are sometimes useful.

Since  $XA : Xb :: XB : Xa$ ,  
 $\therefore XA : Ab :: XB : aB$ ,  
 or  $XA : XB :: Ab : aB$ .....(1),  
 and since  $Xb : Xa :: XA : XB$ ,  
 $\therefore Xb : ab :: XA : AB$ ,  
 or  $Xb : XA :: ab : AB$ .....(2);  
 therefore, multiplying (1) and (2),

$$Xb : XB :: Ab . ba : AB . Ba.$$

Again, since  $QbPB$  is harmonic,

$$\therefore Qb : QB :: Pb : PB,$$

$$\text{or } Qb : Pb :: QB : PB,$$

$$\therefore Qb : Pb :: QB : PB,$$

$$\text{or } 2XP : Pb :: 2XB : PB,$$

$$\therefore Pb^2 : PB^2 :: XP^2 : XB^2$$

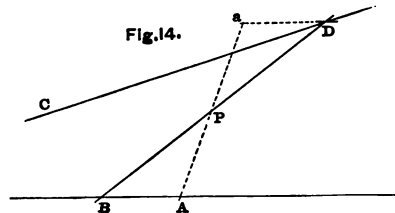
$$:: Xb : XB$$

$$:: Ab . ba : AB . Ba.$$

This determines the ratio in which  $Bb$  is divided by  $P$ .

**PROBLEM 14.** (Fig. 14.) *Through a given point  $P$  to draw a line meeting two given lines  $AB$  and  $CD$  in  $B$  and  $D$  so that  $PB = PD$ .*

Through  $P$  draw any line meeting one of the given lines as at  $A$ . On  $AP$  produced make  $Pa = PA$  and draw  $aD$  parallel to  $BA$

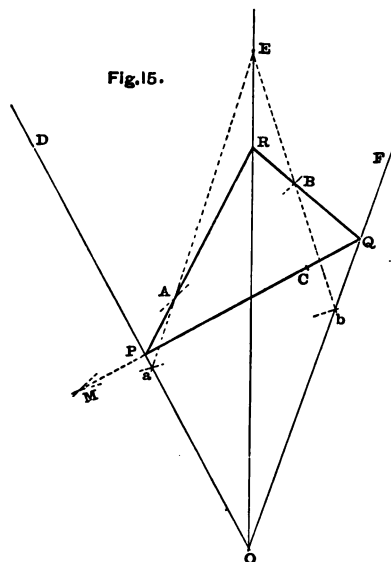


meeting the other given line in  $D$ . The line  $DPB$  will be the line required, i.e.  $PB = PD$  (by the similar and equal triangles  $APB, aPD$ ).

20 TRIANGLE WITH VERTICES ON GIVEN LINES, &C.

**PROBLEM 15.** *To draw a triangle with its sides passing through three given points  $A, B, C$ , and with its vertices on three given concurrent lines  $OD, OE, OF$  (Fig. 15).*

Take any point (as  $E$ ) on any one of the given lines and from it draw lines to any two of the given points (as  $EA, EB$ ) meeting



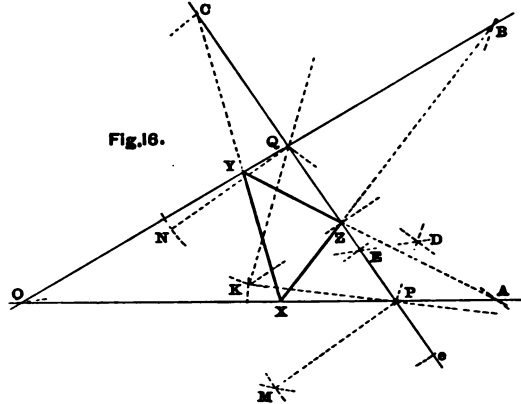
the other lines in  $a$  and  $b$ . Let the lines  $AB$  and  $ab$  meet in  $M$ . Through  $M$  draw a line  $MC$  passing through the remaining point ( $C$ ) and meeting the lines  $Oa$  and  $Ob$  in  $P$  and  $Q$ .  $PQ$  will be one side of the required triangle which can be completed by drawing the lines  $PA, QB$  which will intersect in  $R$  on the third given line.

There are generally six solutions as lines can be drawn through each point terminated by either pair of lines.

**PROBLEM 16.** *To draw a triangle with its vertices on three given lines  $AP, BQ, CQP$ , and with its sides passing through three given points  $A, B, C$  one on each line (fig. 16).*

Let two of the given lines (as  $AP, BQ$ ) meet in  $O$ ; the third line

meets the others in  $P$  and  $Q$ . Draw the lines  $AQ$  and  $BP$  intersecting in  $D$ , and draw  $OD$  intersecting  $PQ$  in  $E$ . Take a mean



proportional  $PM$  between  $CP$  and  $PE$  (Problem 5), and a mean proportional  $QN$  between  $CQ$  and  $QE$ . With centres  $P$  and  $Q$  and radii respectively equal to  $PM$  and  $QN$  describe arcs intersecting in  $K$ . Draw a line bisecting the angle  $PKQ$ , intersecting  $PQ$  in  $Z$ .  $Z$  will be one of the vertices of the required triangle which can be completed by drawing  $BZ$  intersecting  $AP$  in  $X$  and  $AZ$  intersecting  $BQ$  in  $Y$ .  $X$  and  $Y$  are the other vertices and  $XY$  will pass through  $C$ .

**PROBLEM 17.** *To determine the locus\* of the vertex of a triangle on a given base  $AB$  and with sides  $BP$ ,  $AP$  in a given ratio  $a : b$ . (Fig. 17.)*

On the given base  $AB$  describe any one triangle with sides

$$BP : AP :: a : b.$$

Bisect the angle  $APB$  by  $PD$  meeting  $AB$  in  $D$  and draw  $PC$  perpendicular to  $PD$  meeting  $AB$  in  $C$ .

On  $DC$  as diameter describe a circle, which will be the required locus of the vertex.

\* For definition of locus, see p. 29 *post*.







## EXAMPLES ON CHAPTER I.

1. Draw a circle of radius 2.87. In it place a chord  $AB$  of length 4.8, and draw  $BC$  making  $60^\circ$  with  $AB$ . If  $C$  is on the circle shew that the side  $AC$  of the triangle is approximately 4.96.

Shew that the geometrical mean between 3.76 and 2.43 is 3.02 approximately.

2. Inscribe a square in a given triangle  $ABC$ .

(Through  $A$  draw a parallel  $AD$  to  $BC$ ; make  $AD$  equal in length to the perpendicular from  $A$  to  $BC$ , and join  $D$  to the end of the base  $BC$  that will enable it to cut one of the sides  $AB$  or  $AC$  in  $E$ .  $E$  is one of the angular points of the required square, the base of which will coincide in direction with  $BC$ .)

3. Bisect a given triangle  $ABC$  by a straight line drawn through a given point  $D$  in  $AC$ .  $AD < DC$ .

(Bisect  $BC$  in  $E$  and through  $A$  draw  $AF$  parallel to  $DE$  meeting  $BC$  in  $F$ .  $DF$  will be the required line.)

4. Given the middle points  $P$ ,  $Q$ ,  $R$  of the sides of a triangle, construct the triangle.

(The side through  $P$  is parallel to  $QR$ , and so for the others. Take  $PQ = 2$ ,  $QR = 1.8$ ,  $RP = 1.3$ .)

5. Construct a triangle having given the base  $AB$ , the vertical angle  $C$ , and the difference of the sides  $AC$ ,  $CB$ .

(Construct a triangle  $ADB$  having the angle  $ADB = 90^\circ + \frac{C}{2}$ ,  $DA =$  the given difference and  $AB$  the given base. Produce  $AD$  to  $C$ , and make the angle  $DBC =$  the angle  $BDC$ .)

6. Construct a triangle, being given the base  $AB$  the difference of the base angles, and the difference of the sides  $AC$ , and  $BC$ .

(Make a triangle  $DBA$ , with angle  $DBA = \frac{1}{2}$  the given differ-

ence.  $BA$  = the given base and  $AD$  the given difference of sides : produce  $AD$  to  $C$  and make the angle  $DBC = \text{angle } BDC$ .)

7. Construct a triangle, being given the base  $AB$ , the vertical angle  $C$  and the sum of the sides  $AC$  and  $BC$ .

(Make an angle  $ADB = \frac{C}{2}$ , make  $DA = AC + BC$ , and  $AB$  = the given base; make the angle  $DBC = \frac{C}{2}$ , and so that  $BC$  cuts  $AD$  in  $C$  between  $A$  and  $D$ .)

8. Let  $ABC$  be any triangle,  $CD$  a perpendicular from  $C$  on  $AB$  and  $E$  a point on  $AB$  such that  $DE = DB$ .  $AE$  is the difference of the segments of base made by the perpendicular, then given  $AE$  and any one of the following pairs of data, construct the triangle.

$\alpha$ . Sum of sides ( $AC + BC$ ) and difference of base angles.

(We are given in the triangle  $ACE$ ,  $AC + CE$ ,  $AE$  and vertical angle  $ACE$ , i. e. base, vertical angle and sum of sides. The triangle can therefore be constructed (last example) and from it the required triangle  $ABC$ .)

$\beta$ . Difference of sides and difference of base angles.

(Make an angle  $ADE$  containing  $90^\circ + \frac{\alpha}{2}$  where  $\alpha$  is given difference. Make  $DA$  = the given difference of sides, and  $AE$  the given difference of segments; produce  $AD$  to  $C$  and make  $DEC = EDC$ ; produce  $AE$  to  $B$  and make  $CB = CD = CE$ .  $ABC$  will be the required triangle.)

$\gamma$ . Sum of sides and vertical angle.

(Construct a triangle  $AEC$  on the given difference of segments  $AE$  as base, with  $AC + CE$  = given sum of sides and the given vertical angle as difference of base angles ( $\alpha$  above), produce  $AE$  to  $B$  and make  $CB = CE$ .)

$\delta$ . Difference of sides and vertical angle.

(Make an angle  $AEF$  = half the given angle, make  $AF$  = the given difference of sides, produce  $AF$  to  $C$  and make the angle  $FEC = EFC$ , produce  $AE$  to  $B$  and make  $CB = CE = CF$ .)

9. Given the lengths  $AD$ ,  $BE$ ,  $CF$  of the bisectors of the sides of a triangle  $ABC$ , to construct the triangle.

(Construct a triangle  $FOG$  making  $FG = \frac{1}{3}AD$ ,  $GO = \frac{1}{3}BE$ ,  $FO = \frac{1}{3}FC$ ; produce  $FO$  to  $C$  and make  $OC = 2.OF$  so that  $FC$  is the given length; produce  $GO$  both ways to  $B$  and  $E$  and make  $BG = OE = GO$  so that  $BE$  is the given length. Join  $BC$  and draw  $BF, CE$ , producing them to meet in  $A$ .  $ABC$  will be the required triangle.)

10. Given the lengths  $AD$ ,  $BE$ ,  $CF$  of the perpendiculars on the sides from the opposite angles of a triangle  $ABC$ , to construct the triangle.

(Determine a length  $Mb$  such that  $CF : BE :: AD : Mb$ , and on it construct a triangle  $Mbc$ , making  $bc = BE$  and  $Mc = AD$ . From  $M$  drop a perpendicular on  $bc$  and on it make  $Md = AD$ . Through  $d$  draw  $BdC$  parallel to  $b$ , meeting  $Mb, Mc$  in  $B$  and  $C$ .  $MBC$  will be the required triangle.)

11. Given three points  $D, E, F$ , to construct a triangle of which these points shall be the feet of the perpendiculars on the sides from the opposite angles.

(The sides are perpendicular to the bisectors of the angles of the triangle  $DEF$ .)

12. Divide a given straight line  $AB$  into two parts  $AC, CB$ , such that the difference of the squares on the parts may be equal to the square on a given line  $DE < AB$ .

(Take a third proportional  $FG$  to  $AB$  and  $DE$ .  $FG$  will be the difference between the required parts, and  $AB$  is their sum, so that  $AC$ , and  $CB$  are known.)

13. Divide a given line  $AB$  into two parts  $AC, CB$ , such that the square on  $AC$  may be double the square on  $CB$ .

(Take  $AC : CB :: \sqrt{2} : 1$ .)

14. Divide a given straight line  $AB$  into two parts  $AC, CB$ , such that the sum of their squares shall be equal to the square on a given line  $DE$ .

$$DE > \frac{AB}{\sqrt{2}} < AB.$$

(Construct a rectangle equal in area to  $2DE^2 - AB^2$  (Prob. 18). Take a mean proportional between its sides which will be the difference between  $AC$  and  $CB$ ; the sum and difference of the parts being known, the parts are known.)

15. Divide a given straight line  $AB$  into two parts  $AC$ ,  $CB$ , such that  $AB^2 + CB^2 = 2.AC^2$ .

(Take  $AC : CB :: 1 + \sqrt{3} : 1$ .)

16. Draw any triangle  $ABC$ , bisect  $AB$  in  $D$ , join  $CD$ , and through  $C$  draw  $CE$  parallel to  $AB$ ; shew by drawing a transversal, that the rays  $CA$ ,  $CD$ ,  $CB$ ,  $CE$  form a harmonic pencil.

17. Given the directions of one pair of opposite sides of a quadrilateral  $AB$  and  $CD$ , and the point ( $F$ ) of intersection of the other pair, shew that the locus of the intersection of the diagonals is a straight line.

(If  $AB$ ,  $CD$  intersect in  $F$ , and  $G$  is the intersection of the diagonals, the pencil  $E(AGCF)$  is harmonic.)

18. Find the geometric mean ( $BD$ ) between two given lines ( $AB$  and  $BC$ ) and shew by construction that the harmonic mean between  $AB + BD$ , and  $BC + BD$  is  $2BD$ .

19. A line  $AB$  is divided harmonically in  $C$  and  $D$ , and a part  $CB$  of the line which contains two terms  $CD$  and  $DB$  is bisected in  $E$ . Shew that  $EC$  is the geometric mean of  $EA$  and  $ED$ .

20. Divide a given straight line  $AB$  medially in the point  $C$ , and produce the line so that the part produced is equal to  $AC$  the smaller segment; shew by construction that the rectangle contained by  $AC$  and the whole line thus produced, together with the square on  $AB$  is equal to four times the square on  $CB$ .

## CHAPTER II.

### THE CIRCLE.

EUCLID's well known definition is "A circle is a plane figure contained by one line, which is called the circumference, and is such, that all straight lines drawn from a certain point within the figure to the circumference are equal to one another : and this point is called the centre of the circle". A radius of a circle is a straight line drawn from the centre to the circumference, and therefore by the above definition all radii of a circle are equal.

Hence a circle is completely determined if we know its centre and the length of its radius, and it might seem at first sight that *two* geometrical conditions would be sufficient to determine it. The position of the centre however must be counted as two conditions, and a circle can generally be drawn to satisfy *three* geometrical conditions, and three are in general necessary and sufficient for its determination. Thus an infinite number of circles can be drawn to pass through two points, or to touch two lines, and some other condition, such as the position of a third point through which it must pass or of a line which it must touch in the first case, or of a third line which it must touch or of a point through which it must pass in the second, or such as the length of the radius in either, must be given to make the exact solution of the problem possible.

The above limitation "in general" is necessary because it is possible to give certain special positions to the lines and points which would render the problem impossible : thus e.g. in the first case a circle cannot be drawn through three points in the same straight line, or at least no circle of finite radius, or if the given conditions are "to pass through two given points and touch a

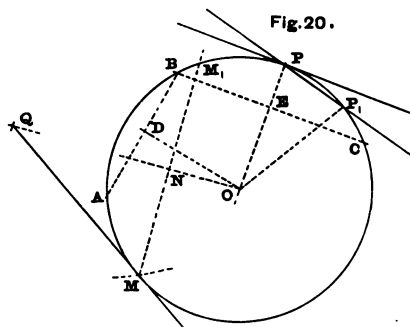


iven line" the line must obviously lie outside the points, i.e. it must not pass between them, and similarly if the conditions are 'to touch three given lines' one at least of the lines must not be parallel to the other two, but notwithstanding these special cases it is generally true that a circle can be drawn to satisfy any *three* geometrical conditions.

**DEFINITION.** When a point is restricted by conditions of any kind, to occupy any of a particular series of positions, that series of positions is called the *locus of the point*.

**PROBLEM 20.** (Fig. 20.) *To describe a circle through three given points  $A, B, C$ , not in the same straight line.*

If the line joining  $A, B$  is bisected in  $D$  and  $DO$  is drawn perpendicular to  $AB$ ,  $DO$  will obviously be the locus of the centres of



all circles passing through  $A$  and  $B$ , i.e. any circle through  $A$  and  $B$  must have its centre on  $DO$ , since in the equal right-angled triangles  $ADO, BDO$ ,  $AO$  is equal to  $BO$ . Similarly, bisecting  $BC$  in  $E$  and drawing  $EO$  perpendicular to  $BC$ ,  $EO$  is the locus of centres of circles passing through  $B$  and  $C$ . Hence the centre of the circle passing through  $A, B$  and  $C$  must lie simultaneously on both these loci, i.e. must be at their intersection, and the distance from this point to either  $A, B$  or  $C$  will be the radius of the required circle.

Euclid in definition 2 of Book III. defines a tangent to a circle in these words. "A straight line is said to touch a circle when it



meets the circle, and being produced does not cut it," and shews in Corollary to Prop. 16, Bk. III. that the line drawn perpendicular to a radius at its extremity fulfils the condition of this definition. This is the most convenient way in which to draw the tangent at any point on the circumference, and the tangent so drawn can easily be shewn to agree with the general definition of a tangent usually given as applicable to all curves, which is as follows:—

DEFINITION. If two points be taken on a curve and a chord drawn through them; then, if the first point remains fixed while the second, moving along the curve, approaches indefinitely near to the first, the chord in its limiting position is called the tangent to the curve at the first point.

To shew that such chord in its limiting position will in the circle be perpendicular to the radius at the point, take two points  $P, P_1$ , (fig. 20) on the curve, and draw  $PP_1$ , then since  $OP = OP_1$  the angles  $OPP_1$  and  $OP_1P$  are equal and will remain equal however close  $P_1$  may be taken to  $P$ . But when  $P_1$  coincides with  $P$  each of these angles becomes a right angle, i.e. the tangent at  $P$  will be perpendicular to  $OP$ .

To draw a tangent to the given circle from an external point  $Q$ . Join  $OQ$  and on it as diameter describe a circle cutting the given circle in  $M$  and  $M_1$ . (It will necessarily do so in two points on opposite sides of its diameter.)

Then  $QM, QM_1$  will be tangents to the circle since  $QMO$  is a right angle being in a semicircle. (Euclid, Prop. 31, Bk. III.) It is always possible to draw two tangents to a circle from any external point.

#### POLE AND POLAR. (Fig. 20.)

The line  $MM_1$  is evidently perpendicular to  $OQ$  for the triangles  $QOM, QOM_1$  are equal in all respects, i.e. the angle  $MOQ$  = the angle  $M_1OQ$ ; then if  $MM_1$  meets  $OQ$  in  $N$  we have in the two triangles  $NOM, NOM_1$ ,  $OM = OM_1$ ,  $ON$  common and the angle,  $NOM$  = the angle  $NOM_1$ ,  $\therefore$  the angle  $ONM$  = the angle  $ONM_1$ , and  $\therefore$  each is a right angle.

The triangle  $MON$  is  $\therefore$  similar to the triangle  $QOM$  and

$$\therefore ON : OM :: OM : OQ,$$

or

$$ON \cdot OQ = r^2,$$

where  $r$  is the radius of the circle.

Now whether the point  $Q$  be taken inside or outside the circle, it is always possible to find on the line  $OQ$  a point  $N$  fulfilling the above condition, and a line  $MNM_1$  drawn perpendicular to  $OQ$  through the point  $N$  so determined is called the *polar* of  $Q$  with respect to the circle, while the point  $Q$  is called the *pole* of  $MM_1$  with respect to the circle.

To draw the *polar* of any point  $Q$  with respect to a given circle.

If the given point be without the circle the polar is, by the previous definition, the *chord of contact* of the tangents drawn from  $Q$  to the given circle. If the given point be within the circle, draw  $OQ$  and produce it, and through  $Q$  draw  $MQM_1$  perpendicular to  $OQ$ , and meeting the circle in  $M$  and  $M_1$ , and at either  $M$  or  $M_1$  draw  $MN$  or  $M_1N$  a tangent to the circle, meeting  $OQ$  produced in  $N$ , then a line through  $N$  perpendicular to  $OQ$  will be the required *polar*.

*Cor. 1.* If the given point be *on* the circle its polar is the tangent at the point, i.e. the *polar* passes through the *pole*.

*Cor. 2.* If a point  $A$  lie on the polar of  $Q$  then  $Q$  lies on the polar of  $A$ . For draw  $OA$  and on it drop a perpendicular  $Qq$  from  $Q$  meeting it in  $q$  and the circle in  $m$  and  $m_1$ ; then the triangles  $OQq$ ,  $oAN$  are similar and

$$\therefore Oq : OQ :: ON : OA, \text{ i.e. } Oq \cdot OA = OQ \cdot ON = r^2,$$

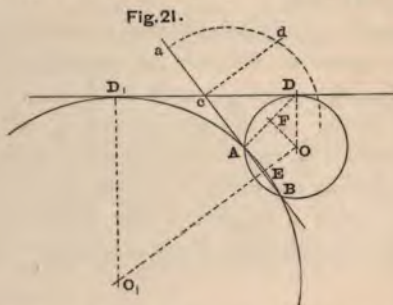
by definition,  $r$  being the radius of the circle, i.e.  $mQm_1$  is the polar of  $A$  which consequently passes through  $Q$ .

*Cor. 3.* The pairs of tangents drawn at the extremities of *any chord* through  $Q$  intersect in the straight line  $AB$  the polar of  $Q$ . Hence the *polar* may be defined as the locus of the points of intersection of tangents at the extremities of chords through a fixed point.

Given a circle and a triangle  $ABC$ , if we take the polars with respect to the circle, of  $A, B, C$ , we form a new triangle  $A'B'C'$  called the *conjugate* triangle,  $A'$  being the pole of  $BC$ ,  $B'$  of  $CA$ , and  $C'$  of  $AB$ . In the particular case where the polars of  $A, B, C$  respectively are  $BC, CA, AB$ , the second triangle coincides with the first, and the triangle is called a *self-conjugate* triangle.

PROBLEM 21. (Fig. 21.) *To describe a circle to pass through two given points and touch a given straight line, lying outside the points.*

Let  $A$  and  $B$  be the given points and  $DD_1$  the given straight line. It will be observed that the point of contact of the line is



not given—this would be a fourth geometrical condition and therefore if a circle is required to *touch a given line at a given point*, it can only in general fulfil one other condition as e.g. pass through one point outside the line. See next problem.

Join  $AB$  and produce it to cut  $DD_1$  in  $C$  and indefinitely beyond as to  $a$ . It is a known proposition (Euclid 36, Book III.), that “if from any point without a circle two straight lines be drawn, one of which cuts the circle, and the other touches it; the rectangle contained by the whole line which cuts the circle and the part of it without the circle, shall be equal to the square on the line which touches it.”

If therefore a mean proportional  $cd$  be taken between  $CA$  and  $CB$ , and its length be set off from  $C$  along  $DD_1$  as  $CD$ , then obviously a perpendicular to  $DD_1$  through  $D$  will be the locus of the centres of circles touching the given line in  $D$ . If  $AB$  be

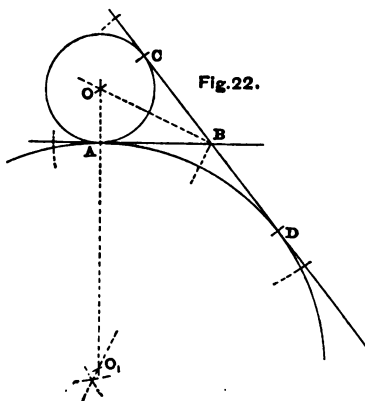
bisected in  $E$ , and  $EO$  be drawn perpendicular to  $AB$ ,  $EO$  will be the locus of centres of circles through  $A$  and  $B$ . The required centre will therefore be at  $O$ , the intersection of these loci and the distance to  $A$ ,  $B$ , or  $D$  will be the required radius. Since the length  $CD$  may be set off on either side of  $C$  there are obviously two solutions as shewn.

If the line joining  $A$  and  $B$  be parallel to the given line, this solution fails, but the point of contact can be at once determined, since by symmetry it is obviously where  $EO$  cuts the given line, and a third point through which the circle must pass being thus obtained the solution can be completed by Problem 20.

**PROBLEM 22.** (Fig. 21.) *To describe a circle to pass through a given point  $A$  and to touch a given straight line  $DD_1$  in a given point  $D$ .*

The straight line  $DO$  through  $D$  perpendicular to  $DD_1$  is obviously the locus of centres of all circles touching the straight line in  $D$ , and the straight line  $FO$  through  $F$  the centre point of  $AD$ , perpendicular to  $AD$  is the locus of centres of all circles through  $A$  and  $D$ . The centre of required circle is therefore at  $O$ , the intersection of these loci, and the distance from  $O$  to  $A$  or  $D$  will be the required radius.

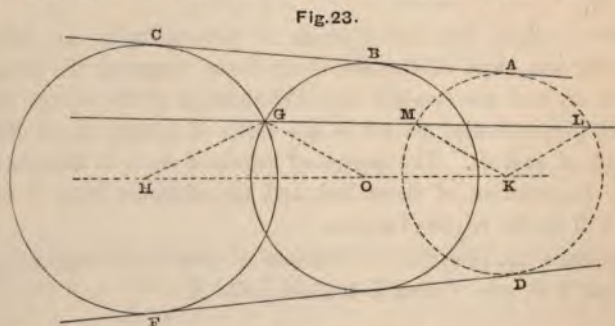
**PROBLEM 23.** (Fig. 22.) *To describe a circle to touch two straight lines  $AB$ ,  $CD$ , one of them in a given point  $A$ .*



A locus of the centre is of course the line  $AO$  perpendicular to  $AB$ . A second locus will obviously be the line  $BO$  bisecting the angle  $ABC$ , and the required centre will therefore be at  $O$ . If at  $O$  a perpendicular  $OC$  be drawn to  $BC$ , the triangles  $OBA$  and  $OBC$  are equal in all respects and therefore  $OA = OC =$  the required radius. The given lines make with each other the angle  $ABD$  as well as the angle  $ABC$ , and therefore bisecting the angle  $ABD$ , the centre  $O_1$  of a second circle is obtained touching the other side of  $AB$ .

PROBLEM 24. (Fig. 23.) *To describe a circle to pass through a given point  $G$  and to touch two given lines  $AC$ ,  $DF$ .*

The centre must obviously lie on the line  $KH$  bisecting the angle between the given lines in which the given point lies.

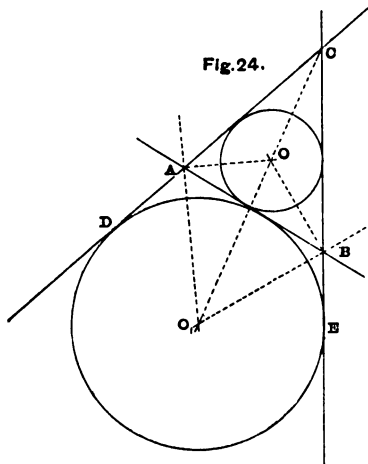


(See Problem 1.) Draw also the line  $GL$  passing through  $G$  and the intersection of the given lines (Problem 6). Take any point  $K$  on  $HK$  as centre, and describe a circle touching  $AC$  and  $DF$ , and cutting  $GL$  in  $M$  and  $L$ . This is always possible, since  $KH$  is the bisector of the angle between these lines. Draw  $GO$  parallel to  $MK$  and  $GH$  parallel to  $KL$ ,  $O$  and  $H$  will be centres of circles fulfilling the required conditions: for if  $A$  is the point of contact of the trial circle,  $KA$  will be perpendicular to  $AC$ , and if  $OB$ ,  $HC$  be drawn perpendicular to  $AC$ ,  $KA$ ,  $OB$  and  $HC$  will all be parallel, and therefore the triangle  $GOB$  will be similar to  $MKA$  and  $GHC$  to  $LKA$ , but the triangles  $MKA$ ,  $LKA$  are isosceles, therefore also  $OG$  must be equal to  $OB$  and  $HG$  to  $HC$ .



**PROBLEM 25.** (Fig. 24.) *To describe a circle to touch three given lines  $AB$ ,  $BC$ ,  $CA$ , not more than two of which are parallel.*

The line  $AO$  bisecting the angle  $BAC$  will be the locus of centres of circles touching  $BA$  and  $CA$ , the line  $CO$  bisecting the



angle  $BCA$  will be the corresponding locus for  $BC$  and  $CA$ . Hence  $O$  will be a centre for a circle touching all three lines. Since  $BA$  makes with  $BC$  and  $CA$  not only the angles  $ABC$ ,  $BAC$  respectively, but also the angles  $ABE$ ,  $BAD$ , a second solution is obviously obtained by bisecting the exterior angle  $BAD$  as shewn by  $AO_1$ , and similarly for the remaining sides. Hence four circles can be drawn touching three straight lines. The exterior circles are said to be *escribed* to the triangle  $ABC$ .

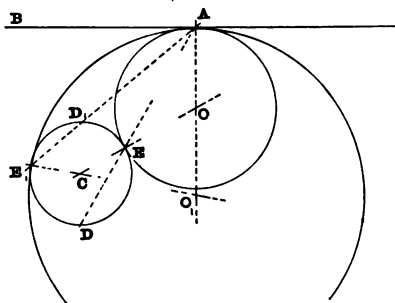
**PROBLEM 26.** (Fig. 25.) *To describe a circle to touch a given circle (centre  $C$ , radius  $CD$ ) and a given straight line  $AB$  in a given point  $A$ .*

The line  $AO$  drawn through  $A$  perpendicular to  $AB$  is a locus of the required centre. Draw a diameter  $DD_1$  of the circle parallel to  $AO$ . Join  $AD$  cutting the given circle in  $E$ , and join

### 36 CIRCLE TO PASS THROUGH TWO POINTS AND TOUCH A CIRCLE.

$CE$  producing it to cut  $AO$  in  $O$ ,  $O$  will be the centre of a circle fulfilling the given condition. A second solution is possible, since

Fig. 25.



$A$  may be joined to either extremity of  $DD_1$ .  $O_1$  is the centre of a second circle.

*Proof.* The angle  $OA E = \text{angle } CDE$ , (Euc. I. 29.)

„  $CED = \text{ „ } CDE$ , (Euc. I. 5.)

„  $CED = \text{ „ } OEA$ , (Euc. I. 15.)

$\therefore \text{ „ } OAE = \text{ „ } OEA$ ,

and  $\therefore OE = OA$ . (Euc. I. 6.)

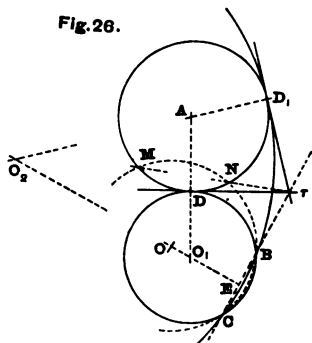
Hence a circle through  $A$  from centre  $O$  will pass through  $E$  and will there touch the given circle, since they will have a common tangent perpendicular to  $CO$ .

**PROBLEM 27.** (Fig. 26.) *To describe a circle to touch a given circle (centre  $A$ , radius  $AD$ ) and pass through two given points  $B$ ,  $C$ , which must be either both inside, or both outside the circle.*

Draw a line through  $BC$ ; bisect  $BC$  in  $E$  and draw  $EO$  perpendicular to  $BC$ .  $EO$  is the locus of centres of circles through  $B$  and  $C$ . Take any point  $O$  such that a circle described with centre  $O$  and radius  $OB$ , or  $OC$  will cut the given circle as in  $MN$ . Draw a line through  $MN$  cutting  $BC$  in  $T$ , and from  $T$  draw tangents  $TD$ ,  $TD_1$  to the given circle (Prob. 20). Lines joining  $AD$ ,  $AD_1$  will cut  $EO$  in points  $O_1$ ,  $O_2$  which will be the centres of circles fulfilling the required conditions.

Two circles can generally be drawn. If the line joining the given points lie wholly without the given circle, one circle will

**Fig. 26.**



touch the given circle externally and one internally (as in the fig.); if the line joining the points cut the given circle, and both points lie on the same side of the circle, both circles will touch the given circle externally, and if the points lie on opposite sides of the circle both will touch it internally. If the line joining the given points touch the given circle one circle only can be drawn.

*Proof.* The rectangle  $TM \cdot TN = \text{rect. } TB \cdot TC$  (Euc. III. 36, Cor.),

„ „ = sq. on  $TD$ . (Euc. III. 36).

$\therefore$  sq. on  $TD = \text{rect. } TB.TC$

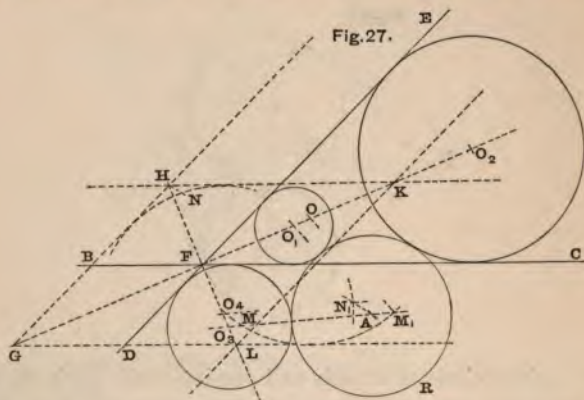
$\therefore TD$  is a tangent to the circle going through  $B, D, C$ .

**PROBLEM 28.** (Fig. 27.) *To describe a circle to touch a given circle (centre  $A$ , radius  $AR$ ) and two given straight lines  $BC$ ,  $DE$ .*

There are several solutions depending on the relative positions of the lines and circle. If the lines are parallel the problem is impossible unless some part of the circle lies between the lines. In this case the line drawn midway between the lines parallel to either of them is evidently a locus of the required centre; a second locus will be the circle described with centre  $A$  and radius equal to the sum of  $AR$  and half the distance between the lines, and since these loci intersect in two points, either may be taken as



the centre of the required circle. If the given lines are not parallel and the given circle cuts one of them, as in the fig., then by



drawing lines parallel to the given lines at a distance from them equal to the radius of the given circle, the problem may be reduced to describing a circle to touch these lines (in pairs) and to pass through the centre of the given circle, i.e. may be reduced to Problem 24. Let the given lines intersect in  $F$  and consider first the circles which can be drawn in the angle  $EFC$ . Draw  $GH$ ,  $LK$  parallel to  $FE$  at a distance from it equal to  $AR$  and similarly  $HK$  and  $GL$  parallel to  $FC$ .  $FK$  will bisect the angle  $CFE$  and will be the locus of the required centres. Take any point  $O$  on it as centre and describe a circle to touch  $GH$  and  $GL$  cutting  $GA$  in  $M$ ,  $M_1$ . Then  $AO_1$  drawn parallel to  $M_1O$  to cut  $FK$  in  $O_1$  determines  $O_1$  a required centre and  $AO_2$  parallel to  $MO$  determines  $O_2$  a second required centre. Similarly for the circles lying in the angle  $DFC$ . Any point  $O_4$  on  $FL$  being taken as centre and a circle described to touch  $GH$  and  $HK$  cutting  $HA$  in  $N$  and  $N_1$ ;  $AO_3$  parallel to  $N_1O_4$  determines  $O_3$ , the centre of a third circle fulfilling the required conditions and a line through  $A$  parallel to  $NO_4$  would determine a fourth centre. It is of course accidental that in the figure  $O_3$  falls nearly on  $GA$ .

If the given circle did not cut either of the given lines, it

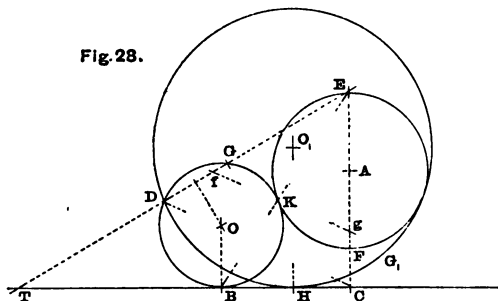
would still be possible to draw four circles touching the lines and the circle, but two of them would have internal contact with the given circle, instead of all touching it externally as in the figure.

If the given circle cut *both* lines there would be six possible solutions.

**PROBLEM 29.** (Fig. 28.) *To describe a circle to touch a given circle (centre  $A$ , radius  $AF$ ) to touch a given line  $BC$  and to pass through a given point  $D$ .*

If the given point be within the circle, the given line must not be wholly outside the circle.

From  $A$  draw  $AC$  perpendicular to the given line and meeting the circle in  $E$  and  $F$ . First join  $ED$  and on it determine a point



*G* such that the rect.  $ED \cdot EG = \text{rect. } EC \cdot EF$ , i. e. take  $EG$  a fourth proportional to  $ED, EC, EF$ . [Making  $Ef$  (on  $ED$ )  $= EF$ , draw  $fg$  parallel to  $DC$  meeting  $EC$  in  $g$  and make  $EG = Eg$ .] Then a circle through  $D$  and  $G$  and touching the given line will also touch the given circle and the problem is reduced to Problem 21. If  $ED, BC$  intersect in  $T$ , a mean proportional ( $TB$ ) must be taken between  $TG$  and  $TD$  so determining the point of contact.  $TB$  may be set off along  $BC$  on either side of  $T$  and hence there are two solutions giving external contact. Second.—Join  $FD$  and on it determine a point  $G_1$  such that rect.  $FD \cdot FG_1 = \text{rect. on } FC, FE$ . i. e. take a fourth proportional to  $FD, FC, FE$ .  $G_1$  must be taken on the opposite side of  $F$  to  $D$  because  $C$  and  $E$  are on opposite sides of  $F$ . Then circles through  $D$  and  $G_1$  touching the given line

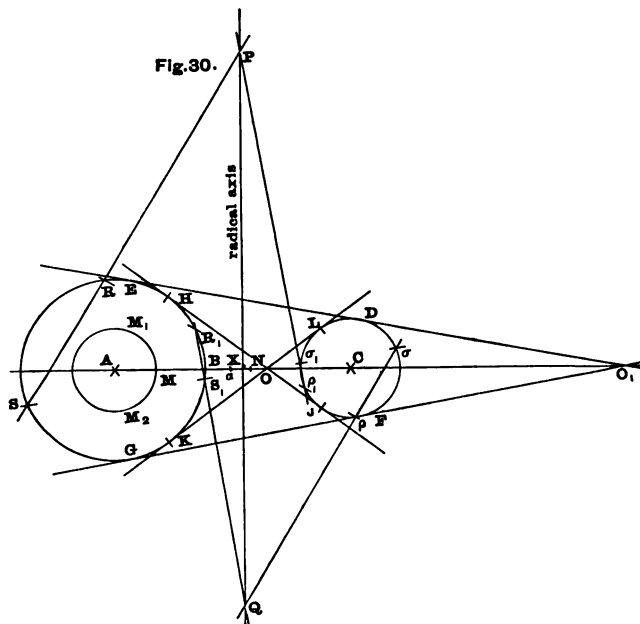


**PROBLEM 31.** (Fig. 30.) *To draw a line touching two given circles, neither of which lies wholly inside the other.*

$A$  and  $AB$  are the centre and radius of the larger circle and  $C$  and  $CD$  those of the smaller circle.

Join  $AC$ , cutting larger circle in  $B$ .

From  $B$  on  $AC$  make  $BM = BN = CD$ , and with  $A$  as centre describe circle  $MM_1M_2$ . From  $C$  draw tangents  $CM_1, CM_2$  to



touch this circle (Prob. 20). Produce  $AM_1, AM_2$  to meet the circle in  $E$  and  $G$ , and lines  $ED, GF$  through  $E$  and  $G$  parallel to  $CM_1, CM_2$  will be tangents to both circles. These tangents meet in  $(O_1)$  a point lying on  $AC$  produced, and are the only pair that can be drawn if the given circles intersect. If the smaller circle lies wholly outside the larger, as in fig. 30, a second pair can be drawn by describing a circle through  $N$  with  $A$  as centre, drawing tangents  $CN_1, CN_2$  to it, and drawing  $HJ, KL$  parallel to these lines re-

spectively, which will intersect in ( $O$ ) a point on  $AC$  between the given circles. The construction is obvious, since  $EM_1 = BM = CD$ .

Common tangents to two circles may be drawn practically with all attainable accuracy by adjusting a set square to touch the circles, and drawing a line by its edge; but the points of contact should always be determined by drawing the radii perpendicular to the tangent.

*Properties of a system of Two or more circles.*

The points  $O, O_1$  in which common tangents to two circles intersect are called the centres of similitude of the two circles. As is easily seen, they are the points where the line joining the centres is cut externally and internally in the ratio of the radii: and in this sense both exist when the circles cut each other, in which case of course only one pair of common tangents can be drawn, and even when one circle lies wholly inside the other, so that it is impossible to draw any common tangent.

If through a centre of similitude we draw any two lines meeting the first circle in the points  $R, R_1, S, S_1$ , and the second in the points  $\rho, \rho_1, \sigma, \sigma_1$ , then the chords  $RS, \rho\sigma$  will be parallel, as also the chords  $R_1S_1$  and  $\rho_1\sigma_1$ ; and the chords  $RS$  and  $\rho_1\sigma_1$ ,  $R_1S_1$  and  $\rho\sigma$  will intersect respectively in points  $P$  and  $Q$  on a line perpendicular to the line joining the centres of the circles.

This line is called the *radical axis* of the two circles.

The rectangle  $OR \cdot OR_1$  is constant, since it equals the square on  $OH$  the tangent from  $O$  (Euc. III. 36), i.e.

$$OR \cdot OR_1 = OS \cdot OS_1$$

and

$$O\rho \cdot O\rho_1 = O\sigma \cdot O\sigma_1.$$

*Proof.* In the triangles  $OAR, OC\rho$ , the angle  $AOR =$  the angle  $CO\rho$  and  $OA : OC :: AR : C\rho$ ,

$\therefore$  also  $OR : O\rho :: AR : C\rho$  (Euc. VI. 7),

i.e. the ratio  $\frac{OR}{O\rho}$  is constant and equal to the ratio of the radii of the circles wherever the line  $OR\rho$  be drawn,

$$\therefore OR : OS :: O\rho : O\sigma$$

and the angle  $ROS =$  the angle  $\rho O\sigma$ ;

$\therefore$  the triangles  $ROS$ ,  $\rho O\sigma$  are similar in all respects, so that the angle  $ORS =$  the angle  $O\rho\sigma$  and  $\rho\sigma$  is parallel to  $RS$ .

Similarly  $R_1S_1$  is parallel to  $\rho_1\sigma_1$ , which proves the first part of the proposition.

Again, since  $SRR_1S_1$  is inscribed in a circle, the angle  $PRO =$  the angle  $SS_1R_1 =$  the angle  $S\sigma_1P$ . The triangles  $PR\rho_1$  and  $P\sigma_1S$  are therefore similar, since the angle  $RP\rho_1$  is common to both.

$$\therefore PR : P\rho_1 :: P\sigma_1 : PS,$$

i.e.

$$PR \cdot PS = P\rho_1 \cdot P\sigma_1,$$

but  $PR \cdot PS =$  square of tangent from  $P$  to circle  $A$ ,

and  $P\rho_1 \cdot P\sigma_1 =$  „ „ „  $C$ ;

$\therefore$  the tangents from  $P$  to the two circles are equal, and

$$\therefore \overline{PA}^2 - \overline{AB}^2 = \overline{PC}^2 - \overline{CD}^2;$$

similarly tangents from  $Q$  to the two circles are equal.

But the locus of the intersection of equal tangents to two circles is a straight line perpendicular to the line joining their centres, and dividing the distance between them so that the difference of the squares of the parts is equal to the difference of the squares of the radii: for if  $X$  be such a point and  $PX$  perpendicular to  $AC$ , at every point on it we shall have

$$PA^2 - AX^2 = PX^2 = \overline{PC}^2 - CX^2,$$

$$\therefore \overline{PA}^2 - \overline{PC}^2 = AX^2 - CX^2;$$

and as above  $PA^2 - PC^2 = AB^2 - CD^2 = AX^2 - CX^2$ .

Hence the line  $PQ$  in the figure must be such locus which proves the second part of the proposition.

**DEFINITION.** A line drawn perpendicular to  $AC$ , the line joining the centres of two given circles, through a point  $X$  on it, such that the difference of the squares of  $AX$  and  $CX$  is equal to the difference of the squares of the radii of the two circles is called the *radical axis* of the two circles.

As already shewn, it is the locus of the intersection of equal tangents to the two circles.



It may be constructed as in the last proposition or immediately from the definition by bisecting  $AC$  in  $a$  (fig. 30), and making  $aX$  towards  $C$ , the centre of the smaller circle, a fourth proportional to  $2AC$ ,  $AB + CD$ , and  $AB - CD$ ,

i.e. by making  $aX : R - r :: R + r : 2AC$ ,

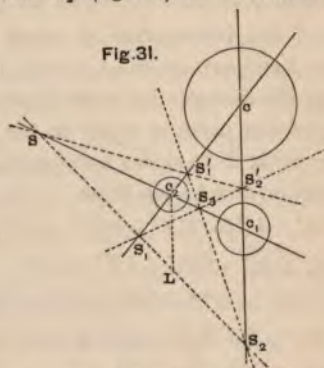
where  $R$  and  $r$  are the radii of the circle, and drawing a line through  $X$  perpendicular to  $AC$ ; for in this case

$$AC \times 2aX = R^2 - r^2 \text{ but } AC = AX + XC \text{ and } 2aX = AX - CX$$

$$\therefore (AX + CX)(AX - CX) = R^2 - r^2 = AX^2 - CX^2.$$

The radical axis bisects the distance between the polars with respect to the two circles, of either centre of similitude, which furnishes another method of constructing it.

Given three circles (centres  $C, C_1, C_2$ , radii  $r, r_1, r_2$ ); the line joining a centre of similitude of  $C$  and  $C_1$  to a centre of similitude of  $C$  and  $C_2$  will pass through a centre of similitude of  $C_1$  and  $C_2$ . Let  $S_2$  and  $S_2'$  (fig. 31) be the centres of similitude of



$C$  and  $C_1$ , and  $S_1$  a centre of similitude of  $C$  and  $C_2$ , and let  $S_2S_1, C_1C_2$  meet in  $S$ ,  $S$  will be a centre of similitude of  $C_1$  and  $C_2$ .

For since  $CS_2' : C_1S_2' :: r : r_1 :: CS_2 : C_1S_2$ ,

$$\therefore CS_2' : CS_2 :: C_1S_2' : C_1S_2,$$

or  $CS_2'C_1S_2$  is a harmonic range; therefore  $S(CS_2'C_1S_2)$  is a harmonic pencil, and therefore if  $CC_2$  cuts  $SS_2'$  in  $S_1'$ ,

$CS_1'C_2S_1$  is a harmonic range, and since  $S_1$  is a centre of similitude of  $C$  and  $C_2$ ,  $S_1'$  must be the other.

Through  $C_2$  draw  $C_2L$  parallel to  $CS_2$  and meeting  $SS_1$  in  $L$ .

Then by similar triangles

$$C_2S : C_1S :: C_2L : C_1S_2,$$

$$\text{and } C_2L : C_2S_1 :: CS_2 : CS_1, \text{ or } C_2L = \frac{C_2S_1 \cdot CS_2}{CS_1},$$

$$\therefore C_2S : C_1S :: \frac{C_2S_1 \cdot CS_2}{CS_1} : C_1S_2,$$

$$\begin{aligned} \text{or } \frac{C_2S}{C_1S} &= \frac{C_2S_1}{CS_1} \cdot \frac{CS_2}{C_1S_2} = \frac{r_2}{r} \cdot \frac{r}{r_1} \\ &= \frac{r_2}{r_1}, \end{aligned}$$

or  $S$  is a centre of similitude of  $C_1$  and  $C_2$ .

Since for each pair of circles there are two centres of similitude, there will be in all *six* for the three circles, and these will be distributed along *four* axes of similitude, as represented in the figure.

*Corollary.* If a circle (centre  $A$ ) touch two others (centres  $C$  and  $C_1$ ) the line joining the points of contact will pass through a centre of similitude of  $C$  and  $C_1$ . For when two circles touch, one of their centres of similitude will coincide with the point of contact. If  $A$  touch  $C$  and  $C_1$  either both externally or both internally, the line joining the points of contact will pass through the *external* centre of similitude of  $C$  and  $C_1$ . If  $A$  touch one externally and the other internally, the line joining the points of contact will pass through the *internal* centre of similitude\*.

Given any three circles, if we take the radical axis of each pair of circles, these three lines will meet in a point, which is called the *radical centre* of the three circles.

For let the radical axes of  $A$  and  $C$  and of  $B$  and  $C$  intersect in  $R$  (fig. 34), then the tangents from  $R$  to  $A$  and  $C$  are equal, as also the tangents from  $R$  to  $B$  and  $C$ ; therefore the tangent from  $R$  to  $A$  must be equal to the tangent from  $R$  to  $B$ , i.e.  $R$  must be a point on the radical of  $A$  and  $B$ , which proves the proposition.

If two circles have a common radical axis, and points  $L$  and  $L_1$  be taken on the line joining their centres at a distance from its

\* Salmon's *Conic Sections*.



intersection ( $X$ ) with the radical axis equal to the tangent which can be drawn from  $X$  to either circle, these points are called the *limiting* points of the entire system of circles which have the same (common) radical axis. They "have many remarkable properties in the theory of these circles, and are such that the polar of either of them, with regard to any of the circles, is a line drawn through the other perpendicular to the line of centres. These points are real when the circles of the system have common two imaginary points, and are imaginary when they have real points common\*."

When they are real it is evidently impossible for the centre of any circle of the system to lie between them, and the more nearly the centre approaches to either of them the smaller must the corresponding radius be. The *limiting* points themselves may therefore be considered as circles of the system of infinitely small radius.

If a system of circles have a common radical axis, and from any point on it tangents be drawn to all the circles, the locus of the points of contact must be a circle, since all these tangents are equal; and it is evident that this circle cuts any of the given system at right angles, since its radii are tangents to the given system. It is the circle passing through the *limiting points* of the system.

Conversely all circles which cut the given system at right angles pass through the limiting points of the system.

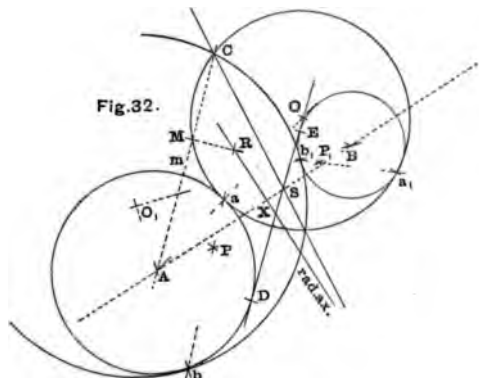
PROBLEM 32. (Fig. 32.) *To describe a circle to touch two given circles (centres  $A$  and  $B$ , radii  $AD$ ,  $BE$  respectively) and to pass through a given point  $C$ .*

Take  $S$  a centre of similitude (p. 42) of  $A$  and  $B$ ; draw  $CS$  and find the poles  $P$  and  $P_1$  of this line with respect to each circle, (i.e. draw  $AP$ ,  $BP_1$  perpendicular to  $CS$  and intersecting the chords of contact of tangents from  $S$  in  $P$  and  $P_1$ ). Draw  $XR$  the radical axis of the given circles (p. 44): draw  $AC$ , bisect it in  $m$  and make  $mM$  on it towards  $C$  of length such that

$$AC : AD :: AD : 2mM;$$

\* Salmon's *Conic Sections*.

draw  $MR$  perpendicular to  $AC$  meeting the radical axis in  $R$ . The lines  $RP, RP_1$  will cut the circles in the points of contact  $a, b; a_1, b_1$  of the required circles and their centres can be at once found by producing  $Aa, Ba$  &c. to meet in  $O$  and  $O_1$ .

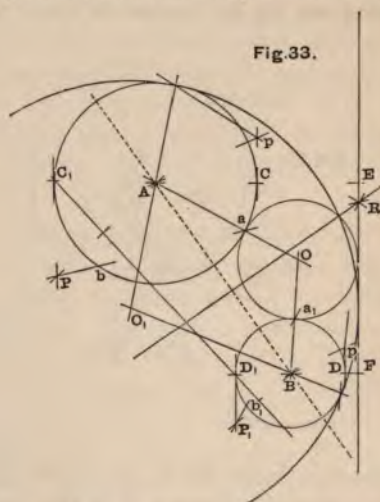


In the figure the circles touch one of the given circles internally and one externally because  $S$  is the internal centre of similitude. If the external one be taken two more circles can be drawn, one touching both externally, the other both internally.

**PROBLEM 33.** *To describe a circle to touch two given circles (centres  $A$  and  $B$ , radii  $AC, BD$  respectively) and a given straight line  $EF$  (fig. 33).*

Draw the radical axis of the given circles, meeting  $EF$  in  $R$ , p. 44. From  $A$  and  $B$  drop perpendiculars on the given line meeting it in  $E$  and  $F$  and the circles in  $C, C_1, D$  and  $D_1$  respectively. Join  $C_1D_1$  cutting  $AB$  in  $S$  a centre of similitude of  $A$  and  $B$ . Find  $P$  and  $P_1$  the poles of this line with respect to the circles (p. 31). Draw  $RP, RP_1$  cutting the circles in  $ab, a_1b_1$ . Then  $aa_1, bb_1$  are the points of contact of circles fulfilling the required conditions, and the intersections of  $Aa, Ba$ , and of  $Ab, Bb_1$  give the corresponding centres. The above circles each touch both of the given circles externally or both internally since  $S$  is the external centre of similitude of  $A$  and  $B$  (p. 45). If  $C_1$  be joined

to  $D$  or  $C$  to  $D_1$  cutting  $AB$  in the internal centre of similitude, the poles of these lines give the points of contact of circles touching one of the given circles internally and the other externally—



and if  $C$  be joined to  $D$  the poles of this line give another pair of circles touching both externally or both internally. One of these latter is shewn in the fig. There are altogether 8 solutions.

**Second Solution.** This problem may also be solved by dropping perpendiculars from  $A$  and  $B$  on the given line as  $AE, BF$ , bisecting the parts lying between the circles and the lines as  $CE, DF$ , in  $G$  and  $H$  and describing parabolas having  $A$  and  $B$  as foci and  $G$  and  $H$  as vertices respectively (Prob. 36). The first will necessarily be the locus of the centres of circles touching the line and the circle  $A$  externally, and the second will be the locus of the centres of circles touching the given line and the circle  $B$  externally, and hence their intersection ( $O$ ) will determine the centre of a circle touching both circles externally and the given line. Similarly if  $C_1E$  be bisected in  $G_1$  and  $D_1F$  in  $H_1$  and parabolas be described having  $A$  and  $B$  as foci and  $G_1, H_1$  as vertices respectively, each of these curves will be the

locus of centres of circles touching the line and the corresponding given circle *internally*. Hence the points of intersection of these four parabolas determine the centres of circles fulfilling the conditions of the problem.

$O_1$  gives internal contact with both circles,

$O_2$  gives internal with  $A$  external with  $B$ ,

$O_3$  gives external or internal        „        „

and so on.

The proof of the construction is obvious from the definition of a parabola subsequently given.

**PROBLEM 34.** *To describe a circle to touch three given circles (centres  $A, B, C$ , radii  $AD, BE, CG$  respectively) (Fig. 34).*

If the circle be required to touch the three either all externally or all internally draw the external axis of similitude  $SS_1$  p. 45,

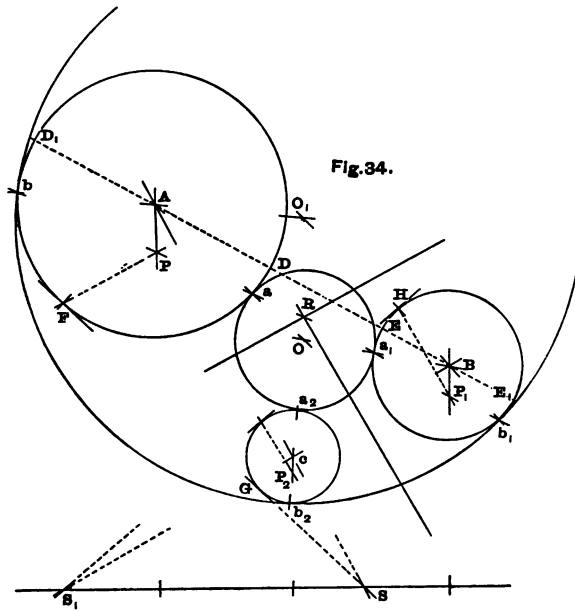


Fig. 34.

and take the poles  $PP_1P_2$  of this line with respect to each circle, p. 31.

Find the radical centre  $R$  of the three circles (p. 45). Then the lines  $RP$ ,  $RP_1$ ,  $RP_2$  cut the circles in the points  $ab$ ,  $a_1b_1$ ,  $a_2b_2$ , in which the required circles must touch them: and the centre of the circle touching all three externally is given by the intersection of  $Aa$ ,  $Ba_1$ ,  $Ca_2$ , which three lines will meet in a point, and the centre of the circle touching all three internally is given by the intersection of  $Ab$ ,  $Bb_1$ ,  $Cb_2$ .

A similar construction with the remaining three axes of similitude, will determine the circles touching one internally and the remaining two externally and vice versa.

There are altogether eight solutions.

Second solution. Join  $AB$  cutting the circles in  $D$ ,  $D_1$ ,  $E$  and  $E_1$ . Bisect  $DE$  in  $K$  and  $D_1E_1$  in  $K_1$ .  $BK$  will necessarily be equal to  $AK_1$ . With  $B$  and  $A$  as foci, and  $K$ ,  $K_1$  as vertices describe an hyperbola (Prob. 89), the branch of which through  $K$  will be the locus of the centres of circles touching circles  $A$  and  $B$  *externally*, and the branch of which through  $K_1$  will be the locus of centres of circles touching these circles *internally*. Similarly, join  $BC$  cutting the corresponding circles in  $F$ ,  $F_1$ ,  $G$ ,  $G_1$ . Bisect  $FG$  in  $L$ , and  $F_1G_1$  in  $L_1$  and with  $C$  and  $B$  as foci, and  $L$ ,  $L_1$  as vertices, describe an hyperbola, the two branches of which will be the loci of centres of circles touching circles  $B$  and  $C$  *externally* and *internally*. The intersection of corresponding branches of the two hyperbolas will therefore determine  $O_1$ ,  $O_2$ , the centres of circles touching the three given circles all *externally* or all *internally*.

Again bisecting  $DE_1$  in  $M$  and  $D_1E$  in  $M_1$  and taking  $B$ ,  $A$  as foci and  $M$ ,  $M_1$  as vertices, an hyperbola can be described the branches of which will be the loci of centres of circles touching circles  $A$  and  $B$ , the one *internally* and the other *externally*, and the intersections of this hyperbola with that through  $L$  and  $L_1$  in  $O_3$ ,  $O_4$  will give centres of two more circles fulfilling the given conditions. The hyperbola through  $N$  and  $N_1$ , points corresponding to

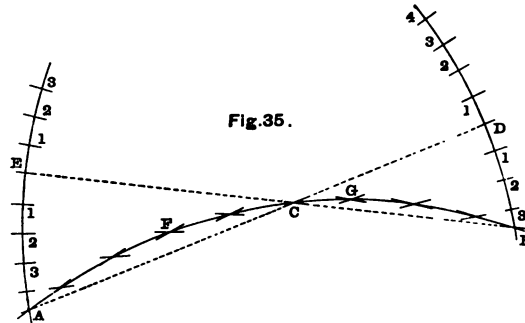


$M, M_1$ , will determine  $O_s$  and  $O_6$ , two additional centres corresponding to  $O_s, O_4$  and lastly, by its intersection with the two branches through  $M$  and  $M_1$  will determine  $O_7$  and  $O_8$ .

The construction is obvious from the definition of the hyperbola subsequently given.

**PROBLEM 35.** (Fig. 35.) *To draw a circular arc through three given points  $A, B, C$  without using the centre.*

Let  $AB$  be greater than either  $AC$  or  $BC$ . With centre  $A$  and radius  $AB$  describe an arc  $BD$  meeting  $AC$  in  $D$ , and with



centre  $B$  and the same radius describe an arc  $AE$  meeting  $BC$  in  $E$ . From  $D$  on each side of it set off on the arc any equal distances  $D1$ , and set off the same distances from  $E$  on the arc  $EA$ , similarly make  $D2 = E2$ , and so on. The line joining  $A$  to any point *above*  $D$  will intersect the line joining  $B$  to the corresponding point *below*  $E$  and vice versa in points (as  $F, G$ ) on the required arc.

*Proof.* It is easily seen that the angle  $AFB = \text{angle } ACB = \text{the angle } AGB$ , &c., and therefore  $AFCGB$  all lie on a circular arc.

## EXAMPLES ON CHAPTER II.

1. Describe a circle to pass through two given points,  $P$  and  $P_1$ , and to bisect the circumference of a given circle (centre  $C$ , radius  $CA$ ).

(Draw  $PC$  and produce it to  $D$  so that  $PC \cdot CD = AC^2$ . The circle through  $P, D, P_1$  fulfils required condition.)

2. Draw two circles cutting orthogonally, and shew by construction that any line through the centre of either cutting both circles is divided harmonically at the points of intersection.

3. Given the base  $AB$  of a triangle and the sum of the squares of the sides  $AC^2 + BC^2$ , draw the locus of the vertex.

(A circle, centre at  $E$  the middle point of  $AB$ , and radius  $= \sqrt{\frac{AC^2 + BC^2}{2} - AE^2}$ .)

4. Draw two circles (centres  $A$  and  $B$ ) cutting orthogonally, and draw their common chord meeting  $AB$  in  $C$ . Draw  $DE$  a chord of the first circle passing through  $B$ , and shew that a circle can be described through  $ADEC$ .

5. The centre  $A$  of a circle lies on another circle which cuts the former in  $B, C$ ;  $AD$  is a chord of the latter circle meeting  $BC$  in  $E$ , shew that the polar of  $D$  with respect to the first circle passes through  $E$ .

6. At two fixed points  $A, B$  are drawn  $AC, BD$  at right angles to  $AB$  and on the same side of it, and of such magnitude that the rectangle  $AC, BD$  is equal to the square on  $AB$ : prove that the circles whose diameters are  $AC, BD$  will touch each other, and that their point of contact will lie on a fixed circle.

(The circle on  $AB$  as diameter.)

7. With three given points  $A, B, C$  not lying in one straight line as centres describe three circles which shall have three common tangents.

(Bisect the angle  $BAC$  by  $AD$  meeting  $BC$  in  $D$ ,

„ „  $CBA$  by  $BE$  „  $CA$  in  $E$ ,

„ „  $ACB$  by  $CF$  „  $AB$  in  $F$ ,

then  $ED, DF, FE$  will be the required common tangents.)

The question is obviously, given the centres of the escribed circles of a triangle, to draw the triangle.

8.  $A$  and  $B$  are two given points on the same side of a given straight line  $CD$ , which  $AB$  meets in  $C$ . Determine the points on  $CD$  on each side of  $C$  at which  $AB$  subtends a greater angle than at any other point on the same side.

(The points of contact of circles through  $A$  and  $B$ , and touching  $CD$ . Prob. 21.)

9.  $A$  and  $B$  are two given points within a circle; and  $AB$  is drawn and produced both ways so as to divide the whole circumference into two arcs. Determine the point in each of these arcs at which  $AB$  subtends the greatest angle.

(The points of contact of circles through  $A$  and  $B$  touching the given circle. Prob. 27.)

10. Shew by construction that the circle which passes through the middle points of the sides of any triangle  $ABC$  will pass through the feet of the perpendiculars from  $A, B, C$  on the opposite sides, and if  $O$  be the intersection of these perpendiculars, will also pass through the middle points of  $OA, OB, OC$ . Shew also that it will touch the inscribed and escribed circles of the triangle, and that its radius is half that of the circumscribing circle.

(The circle is called the nine point circle.)

11. Given four points  $ABCD$  in a straight line taken in order. Shew that the locus of the point  $P$  moving so that the angle  $APB =$  the angle  $CPD$ , is a circle which may be constructed in the following manner. Let  $AB$  be less than  $CD$ , and take  $b$  between  $C$  and  $D$  so that  $bD = AB$ . The centre is on the given straight line at a distance from  $A$ , such that

$$AO : AC :: AB : Cb,$$

and the radius ( $r$ ) is such that

$$r^2 = OB, OC = OA, OD.$$

12. Find the locus of a point such that the area of the triangle whose angular points are the feet of the perpendiculars from it on the three sides of a given triangle, has a constant area.



[It is a circle of radius  $\rho$ , concentric with the circle circumscribing the given triangle; and  $\rho$  is determined from the equation

$$\rho^2 = R^2 \left( \frac{4k}{\Delta} - 1 \right),$$

where  $R$  is the radius of circumscribing circle,  $k$  is the given constant area and  $\Delta$  is the area of the given triangle. If  $4k < \Delta$ ,  $\rho$  is given by the equation

$$\rho^2 = R^2 \left( 1 - \frac{4k}{\Delta} \right).$$

(Salmon's *Conic Sections*, Chap. IX.)]

As a numerical example, draw any triangle  $ABC$ , and take

$$\rho : R :: \sqrt{7} : 1,$$

shew that in this case  $k = \frac{\Delta}{2}$ .

13. Given on a straight line four points in the order  $P, A, B, Q$ ; describe a circle passing through  $A$  and  $B$  such that tangents drawn to it from  $P$  and  $Q$  may be parallel.

[With centres  $P$  and  $Q$  and radii  $\sqrt{PA}, \sqrt{PB}, \sqrt{QA}, \sqrt{QB}$  respectively describe two circles. A circle passing through  $A$  and  $B$ , and through the points of contact of a common tangent to these circles will be the one required.]

14. Given a fixed circle and an external point  $O$ . Draw the tangent at any point  $P$  of the circle and complete the rectangle which has  $OP$  for side and the tangent for diagonal. Shew that the angular point opposite  $O$  will lie on the polar of  $O$ .

15. From the obtuse angle  $A$  of a triangle  $ABC$  draw a line meeting the base in  $D$  so that  $AD$  shall be a mean proportional between the segments of the base.

[Find  $O$  the centre of the circle circumscribing  $ABC$ . On  $AO$  as diameter describe a circle cutting the base in  $D$ , the required point.]

16. Find on a given line  $AB$  a point  $A$  such that its polar with respect to a given circle shall pass through a given point  $C$ .

[Find  $P$  the pole of  $AB$ , then the pole of  $CP$  will lie on  $AB$  i.e. will be the required point  $A$ .]

17. Given a point  $A$ , a line through it  $AB$ , and a circle centre  $C$ ; draw a triangle  $APB$  which shall be *self-conjugate* with respect to the circle (p. 32).

Take  $P$  the pole of the given line and from  $C$  draw  $CB$  perpendicular to  $AP$  meeting  $AB$  in  $B$ ,  $APB$  will be the required triangle; for since  $B$  is on the polar of  $P$  the polar of  $B$  will pass through  $P$ , and is perpendicular to  $CB$ , i.e. is the line  $AP$ .

18. Given a triangle  $APB$  obtuse-angled at  $P$ , to draw the circle with respect to which the triangle shall be self-conjugate.

The centre ( $C$ ) of the circle must evidently be the intersection of the perpendiculars from the angular points on the opposite sides. Let the perpendicular from  $P$  on  $AB$  meet it in  $D$ . The radius of the circle will be a mean proportional between  $CP$  and  $CD$ .

19. Given a circle, describe a triangle which shall be self-conjugate with respect thereto, and with its sides parallel to those of a given triangle  $abp$ , obtuse-angled at  $p$ .

Through  $C$  the centre of the given circle draw  $CA$  perpendicular to  $bp$ ,  $CB$  perpendicular to  $ap$  and  $CM$  perpendicular to  $ab$ . The vertices of the required triangle will lie, one on each of these lines. Through any point  $m$  on  $CM$  draw  $dme$  perpendicular to  $CM$  meeting  $CA$  in  $d$  and  $CB$  in  $e$ , and through  $d$  draw  $df$  perpendicular to  $CB$  and  $Bf$  perpendicular to  $CA$ ;  $f$  will necessarily lie on  $CM$ .

If  $D$  is the point on  $Cm$  through which the side of the required triangle perpendicular to  $Cm$  passes:—

$$|CD|^2 = r^2 \frac{Cm}{Cf},$$

where  $r$  is the radius of the given circle, i.e.  $CD$  is a mean proportional between  $r$  and a length  $l$  determined by taking a fourth proportional to  $Cf$ ,  $Cm$ , and  $r$ ; for if

$$Cf : Cm :: r : l,$$

$$l = r \frac{Cm}{Cf}, \text{ and } \therefore |CD|^2 = lr.$$

## CHAPTER III.

### THE PARABOLA.

IF a line be drawn through the centre of a given circle perpendicular to the plane of the circle, the surface generated by a straight line which passes through a fixed point on the first line and moves round the circumference of the circle is called a right circular cone. It will be shewn in Chap. ix. that the intersection of this surface with any plane must be one or other of the following:—a point, a pair of straight lines, a circle, a parabola, an ellipse or an hyperbola. The construction of these last three curves from their definition as the sections of a cone seems *à priori* to be the natural way of treating the subject; but the fact is they are more easily constructed from some of their known plane properties, and therefore, deferring the consideration of them as lying on the surface of a solid, each will at first be defined as the locus of a point moving in a plane so that its distance from a fixed point is always in a constant ratio to its distance from a fixed line, both point and line being in the plane of motion.

The fixed point is called the focus, and the fixed line the directrix.

In the parabola the ratio is one of equality, i.e. the distance from the fixed point is always equal to the distance from the fixed line.

In the ellipse the ratio is one of less inequality, i.e. the distance from the fixed point is always less than the distance from the fixed line.

In the hyperbola the ratio is one of greater inequality, i.e. the

distance from the fixed point is always greater than the distance from the fixed line.

The eccentricity of a conic is the numerical value of this ratio.

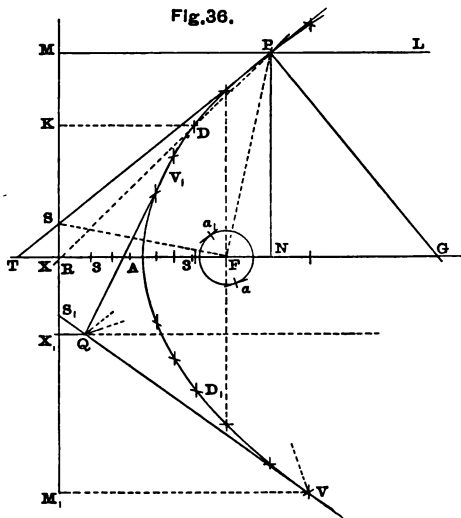
A parabola can generally be drawn to satisfy four geometrical conditions, and four conditions are in general necessary and sufficient to determine the curve. Thus an infinite number of parabolas can be drawn to pass through three given points or to touch three given lines, or to pass through two points and touch a given line, or to fulfil any three similar conditions, and in each case a fourth condition must be added to make the exact solution possible. At the same time four conditions may sometimes lead to more than one solution, just as, more circles than one can frequently be drawn satisfying three given conditions; and occasionally some limitation as to the position of the points or lines given as data of the problem, is necessary to enable a real curve to be drawn.

If the focus is given in any particular problem, this is equivalent to two geometrical conditions, and therefore in general only two others can be fulfilled, i.e. given the focus and two points through which the curve is to pass, the problem is completely determinate and a parabola cannot be drawn to have a given point as focus and to pass through any *three* random points. The directrix being given is also equivalent to two geometrical conditions, and therefore along with it, only two others can be fulfilled, such as, e.g. to pass through a given point and touch a given line, or to touch two given lines, or to touch a given line at a given point, &c.

**PROBLEM 36.** (Fig. 36.) *To draw a parabola the focus  $F$  and the directrix  $MX$  being given.*

Draw  $FX$  from  $F$  perpendicular to  $MX$ . Bisect  $FX$  in  $A$  and  $A$  will be a point in the curve. With  $F$  as centre and any radius greater than  $FA$  (as  $F3$ ) draw a circular arc  $D3D_1$ , set off from  $A$  towards  $F$  a distance  $A3'$  equal to  $A3$ , and at  $3'$  erect a perpendicular to  $FX$  meeting the circular arc in  $DD_1$ . These will be points in the curve, and similarly drawing any number of arcs

with  $F$  as centre and setting off from  $A$  towards  $F$ , distances equal to the distances of the arcs beyond  $A$ , and erecting per-



pendiculars to  $FX$  at these points meeting the corresponding arcs, any number of points on the curve may be determined and the curve drawn through the points thus obtained.

The construction is obvious: at any point as  $P$  draw  $PN$  perpendicular to  $AX$  meeting it in  $N$ ; then the distance  $FP$  from the focus is to be equal to  $PM$  the perpendicular distance from the directrix. But  $FP = FA +$  the distance of the arc beyond  $A$ ; and  $PM = XN = XA +$  the same distance.

$\therefore FP = PM$  since  $FA = AX$ .

(See also the next problem.)

DEF. From the construction the curve is evidently symmetrical about  $FX$  which is called *the axis*. The point  $A$  where the curve cuts the axis is called *the vertex*, and any line parallel to the axis is called *a diameter* of the curve.

The parabola consists of one infinite branch. Like the focus and the directrix, the vertex and axis are each equivalent to two conditions in the construction, but it should be noticed that certain pairs of these lines and points given together are equivalent not to four but only to three conditions. This apparent anomaly may be thus explained. Suppose directrix and axis are given, these are two lines at right angles to each other and hence the direction of either is implicitly involved in that of the other, and thus instead of the two conditions of *position* and *direction* being given independently along with the second line, one only, namely position, is really given, and the two lines together are therefore equivalent to *three* conditions only. Similarly focus and axis, or vertex and axis make only three conditions since the position of the axis is partly involved in that of the focus or of the vertex.

*To draw a tangent at any point.*

$P$  and  $D$  (fig. 36) being any two points on the curve, if the line through  $PD$  meet the directrix in  $R$  and  $DK$  is parallel to  $PM$ , then

$$\begin{aligned} FP : FD &:: PM : DK \\ &:: PR : DR \end{aligned}$$

by similar triangles, and  $\therefore FR$  bisects the exterior angle between  $FP$  and  $FD$  (Euc. vi. Prop. A). Hence if the point  $D$  move up to and coincide with  $P$  so that the chord  $PD$  becomes the tangent at  $P$  (Def. p. 30), in which case  $FD$  of course coincides with  $FP$ , the line  $FS$  drawn from the focus to the point in which the tangent at  $P$  meets the directrix, must be perpendicular to  $FP$ . The triangle  $SFP$  is therefore equal and similar to the triangle  $SMP$ . Hence the tangent at any point  $P$  of a parabola bisects the angle between the focal distance  $FP$  and the perpendicular  $PM$  from  $P$  on the directrix. It can therefore be drawn either by bisecting the angle  $FPM$  or by making  $FT$  on the axis equal to  $FP$ , and joining  $PT$ ; for in this case the angle  $FPT$  = angle  $FTP$ , which is equal to the alternate angle  $TPM$ ,  $PM$ ,  $FT$  being parallel.

**DEF.** The perpendicular  $PN$  from  $P$  on the axis is called the *ordinate* of  $P$ . The double ordinate through the focus is called

the *latus-rectum* of the curve, and its length is always equal to  $4AF$ . It is sometimes called the *principal parameter* of the curve. Since

$$FP = PM = XN = FT \text{ and } FA = AX,$$

$$XN - AX = FT - FA,$$

$$\text{i.e. } AT = AN \text{ or } NT = 2, AN.$$

DEF. The line  $NT$  is called the *sub-tangent* at the point  $P$ .

The line  $PG$  perpendicular to the tangent at  $P$  is called the *normal* at  $P$ .

It has been shewn that the tangent bisects the angle  $FPM$ ,  
 $\therefore PG$  bisects the angle  $FPL$  where  $L$  is a point on  $MP$  produced,

$$\text{i.e. the angle } FPG = \text{angle } LPG = \text{angle } PGF,$$

$$\text{and } \therefore FG = FP = PM = XN,$$

$$\therefore FG - FN = XN - FN,$$

$$\text{i.e. } NG = FX = 2AF.$$

DEF. The line  $NG$  is called the *sub-normal* of the point  $P$ .

The tangent at the vertex is perpendicular to the axis, as is obvious from the symmetry of the curve, and a perpendicular from the focus on any tangent intersects it and the tangent at the vertex in the same point.

The focus and directrix being given, tangents to the curve can be drawn from an external point  $Q$  thus (fig. 36). With centre  $F$  and radius equal to the distance of  $Q$  from the directrix describe a circle; draw tangents to it from  $Q$ , and join  $F$  to the points of contact  $a, a_1$ , producing the lines to meet the curve in  $VV_1$ .  $QV, QV_1$  will be tangents, for, if  $VM_1$  be the perpendicular on the directrix, and the diameter at  $Q$  meet the directrix in  $X_1$  and  $VQ$  meet it in  $S_1$ ,

$$VM_1 : QX_1 :: VS_1 : Q_1S_1,$$

$$\text{or } FV : Fa :: VS_1 : Q_1S_1,$$

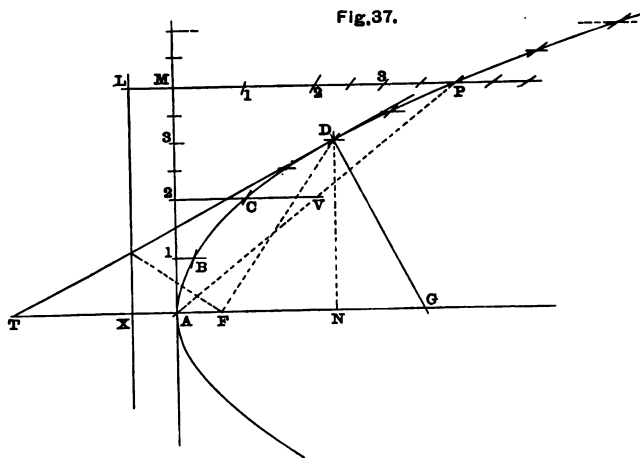
$\therefore FS_1$  is parallel to  $aQ$ , but  $aQ$  is perpendicular to  $FV$ ,  $\therefore FS_1$  is perpendicular to  $FV$ , and  $\therefore VS_1$  or  $VQ$  is a tangent through the point  $Q$ ; similarly for  $V_1Q$ .



A tangent to a parabola parallel to a given line may be drawn by constructing the angle  $GFP =$  twice the angle which the line makes with the axis, so determining the point of contact  $P$ .

PROBLEM 37. (Fig. 37.) To draw a parabola, the vertex  $A$ , the axis  $AN$  and a point  $P$  on the curve being given.

This might be solved by first finding the focus and proceeding as in the last problem. It can however be solved independently



without using circular arcs, and the method is evidently applicable to the last problem after any one point on the curve has been found.

Draw the tangent at the vertex and a diameter through  $P$  meeting it in  $M$ . Divide  $MP$  into any number of equal parts (say four), and  $AM$  into the same number. Then diameters through the several points on  $AM$  will meet lines joining  $A$  to the corresponding points on  $MP$  (counting from  $A$  in the first case and from  $M$  in the second) in points of the curve as  $B, C, D$ . As the curve recedes from the axis the points found get more and more distant from each other (compare  $C$  to  $D$  and  $D$  to  $P$ ), but, if desirable, points can be interpolated between any two points already found by subdividing the corresponding spaces on  $MP$  and



*AM.* In the figure points are thus interpolated between *C* and *D* and between *D* and *P*. The curve can be carried beyond *P* by carrying on the divisions on the two lines as in the figure.

The other half of the curve can be put in by symmetry.

The tangent at any point *D* can be drawn by drawing the ordinate *DN*, and making *AT* on the axis equal to *AN*, on the other side of the vertex; *DT* will be the tangent at *B*, as has already been shewn.

The focus *F* is found by drawing the normal at any point *D*, bisecting the sub-normal *NG* and setting off  $AF = \frac{1}{2}NG$ .

The construction for the curve depends on the fact that if a diameter be drawn through the centre point of any chord, the tangents at the extremities of the chord intersect on the diameter, and the curve cuts the diameter at the centre point between the chord and the intersection of the tangents. Thus *AP* is a chord, the diameter through 2 (on *AM*) will intersect it in its centre point *V*, *A2* is the tangent at *A* and therefore the tangent at *P* will also pass through 2, and *C*, which bisects *V2* since

$$C2 : CV :: M2 : P2$$

will be a point on the curve.

Similarly *B* may be shewn to be on the curve, since it bisects the diameter between 1 and the centre point of the chord *AC*, and *D* may be shewn to be on the curve as bisecting the diameter between *C3* the tangent at *C*, and the centre point of the chord *CP*.

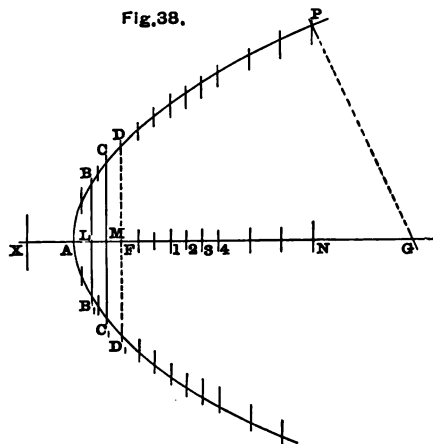
PROBLEM 38. (Fig. 37.) *To draw a parabola the focus F, the axis FN, and a point P on the curve being given.*

The directrix and consequently the vertex can at once be determined by drawing *PM* parallel to the given axis, measuring along it a length *PL* equal to *FP* and from *L* dropping a perpendicular on the axis intersecting it in *X*. This perpendicular is, of course, the directrix, and the vertex bisects *FX*. The curve can then be drawn by either of the preceding methods.

**PROBLEM 39.** (Fig. 38.) *To draw a curve formed of circular arcs approximating to a parabola the focus  $F$ , and vertex  $A$  being given.*

The following method depends on the fact that in the parabola the sub-normal is constant and equal to twice  $AF$ .

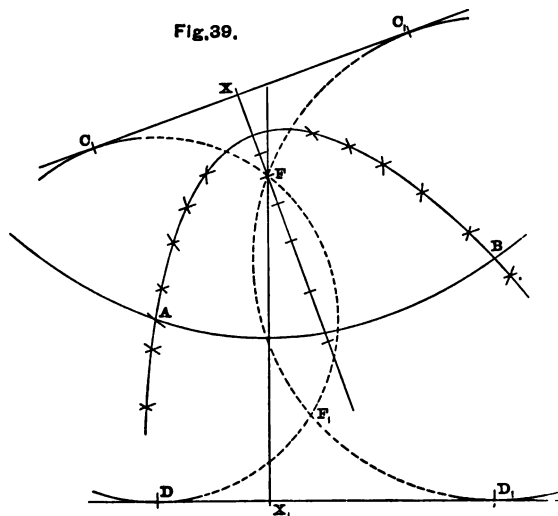
Fig. 38.



Draw the axis  $AFN$  and on it take  $F1$  equal to  $FA$ , and draw any ordinates as  $BB_1$ ,  $CC_1$ ,  $DD_1$ , &c. With 1 as centre describe an arc through  $A$ , extending as far as the centre ordinate between  $A$  and  $BB_1$ , from  $L$  the foot of ordinate  $BB_1$  make  $L2$  equal to twice  $AF$ , and with centre 2 and radius to the point where arc through  $A$  meets the centre ordinate between  $A$  and  $B$  describe an arc extending to half-way between  $B$  and  $C$ ; from  $M$  the foot of ordinate  $CC_1$  make  $M3$  equal to twice  $AF$  and with centre 3 and radius to the point where arc through  $B$  has been stopped describe an arc extending to half way between  $C$  and  $D$ . Similarly from the foot of the ordinate  $DD_1$  measure a distance on the axis equal to twice  $AF$  so determining the centre (4) for an arc through  $D$ , and continue the process for any number of successive ordinates. It will be seen that the centres are determined by measuring a constant distance from the foot of the successive ordinates

equal to the known constant length of the sub-normal in the parabola (p. 60), but that the radius of each arc depends entirely on the arc previously drawn, so that the curve must be commenced from the vertex. Each successive arc extends some distance on each side of the ordinate from which its centre is determined. It is convenient, though not essential, to commence with ordinates dividing  $AF'$  into equal parts, and tolerably close together, and as the curve recedes from the vertex and cuts the ordinates more nearly at right angles the distance between them may be increased. Carefully drawn, the method gives a remarkably close approximation to the real form of the curve, as may be seen by comparing the distance of the point  $P$  in the figure from  $F'$  with the distance  $NX$ , its perpendicular distance from the directrix. The half distance between the ordinates to which each successive arc has to extend, and which furnishes the starting point for the next arc can generally be estimated with quite sufficient accuracy by the eye.

PROBLEM 40. (Fig. 39.) *To draw a parabola, the focus  $F$ , and two points  $A$  and  $B$  on the curve being given.*



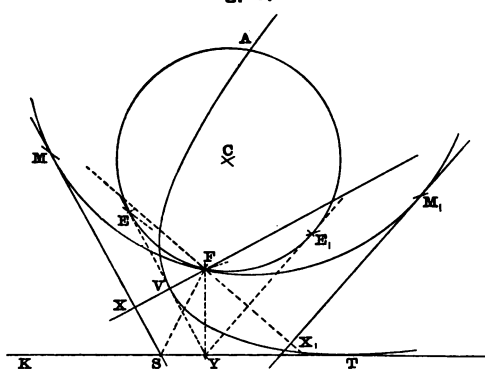
With centre  $A$  and radius  $AF$  describe a circle  $CFD$ , and with centre  $B$  and radius  $BF$  describe a circle  $C_1FD_1$ . Draw common tangents  $CC_1$  and  $DD_1$  to the two circles. (Prob. 31.) These will be the directrices of two parabolas fulfilling the given conditions, and the curves may be drawn by any of the preceding methods.

**The construction is obvious.**

**PROBLEM 41.** (Fig. 40.) *To draw a parabola, the focus  $F$ , a point  $A$  on the curve, and a tangent  $YT$  being given.*

The *point of contact* of the tangent is not given, as this would be a fifth condition. With centre  $A$  and radius  $AF$

**Fig.40.**



describe a circle  $FM$ , and on  $FA$  as diameter describe a circle  $EFE$ , the centre being  $C$ . From  $F$  drop a perpendicular  $FY$  on the given tangent, and from  $Y$  draw tangents  $YE, YE_1$  to the given circle. Join  $FE, FE_1$  and produce them to meet the larger circle in  $M, M_1$ , then  $MX, M_1X_1$  drawn parallel to  $YE, YE_1$  respectively will be the directrices of two parabolas fulfilling the given conditions.

*Proof.* It is known that the perpendicular from the focus on a tangent passes through the point of intersection of that

**E.**

tangent and the tangent at the vertex, hence  $Y$  is a point on the tangent at the vertex.

The directrix must evidently touch the circle  $MF M_1$  and must meet the perpendicular on it from the focus at a point double the distance from  $F$  that it is from the tangent at the vertex.

In the triangles  $AEF$ ,  $AEM$ ,  $AF = AM$ ,  $AE$  is common and the angles  $AEF$ ,  $AEM$  are equal, each being a right angle;

$\therefore FE = EM$  and  $\therefore AM$  is parallel to  $EC$ , since  $FC = CA$ ; but  $EC$  is perpendicular to  $YE$ , and therefore  $MX$  which is parallel to  $YE$  is perpendicular to  $MA$ , and therefore touches the circle  $MF M_1$ .

Draw  $FX$  perpendicular to  $MX$  and let  $YE$  meet it in  $V$ ,

then  $FV : FX :: FE : FM$ ;

$$\therefore FX = 2FV, \text{ since } FM = 2FE.$$

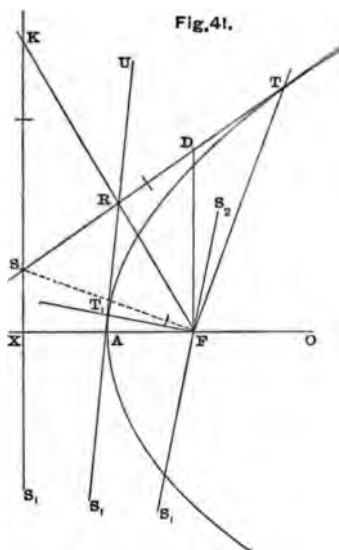
Hence two parallel lines have been found, one of which touches the circle  $MF M_1$ , while the other passes through  $Y$  and bisects the distance between  $F$  and the first.

PROBLEM 42. (Fig. 41.) *To draw a parabola, the focus  $F$  and two tangents  $RT$ ,  $RT_1$  being given.*

(The problem is impossible if the given lines are parallel, i.e. they must always intersect in some point  $R$ ; and  $F$  must not lie on either of them.)

Join  $RF$ , and at  $F$  on each side of  $RF$  construct an angle  $RFT$ ,  $RFT_1$  equal to  $T_1RS$ , the angle between the given lines alternate with that in which  $F$  is situated.  $T$  and  $T_1$  will be the points of contact of the given tangents and the problem is reduced to Problem 40. As in that problem two lines can be drawn touching circles with centres  $T$ ,  $T_1$  and radii  $TF$ ,  $T_1F$  respectively, which will be the directrices of parabolas having  $F$  as focus and passing through  $T$  and  $T_1$ , but only one of these will in addition touch the given lines at those points.

*Proof.* The construction depends on the well-known property of the parabola, that *the exterior angle between any two tangents is*



equal to the angle subtended at the focus, by the segment of either between the point of intersection and the point of contact. For if  $O$  be any point in  $AF$  produced, the angle  $TFO = \text{twice angle } FTR$ , since (Prob. 36) the angle  $FTR = \text{angle which } TR \text{ makes with the axis}$ . Similarly angle  $T_1FO = \text{twice angle } FT_1S_1$ ;

$\therefore \text{angle } T_1FO - \text{angle } TFO = \text{twice } (\text{angle } FT_1S_1 - \text{angle } FTR),$

i.e.  $TFT_1 = \text{twice angle } URT = \text{twice angle } T_1RS.$

Through  $F$  draw  $FD$  parallel to the directrix meeting  $TS$  in  $D$ , then  $FD = FS$ , since  $TS$  bisects the angle between  $FS$  and the directrix. Let  $FR$  meet the directrix in  $K$ .

By similar triangles

$$KS : FD :: KR : FR,$$

$$KS : KR :: FS : FR;$$



and similarly, if  $S_1$  denote the intersection of  $RT_1$  with directrix,

$$KS_1 : KR :: FS_1 : FR,$$

$$\therefore KS : KS_1 :: FS : FS_1.$$

Hence the angles  $KFS$ ,  $KFS_1$  are either equal or supplementary. In the figure they are supplementary, i.e. angle  $KFS$  = angle  $RFS_1$ .

But angle  $RFT$  is the complement of angle  $KFS$

and       "      $RFT_1$        "       "       "        $KFS_1$ ,

$$\therefore \text{angle } RFT = \text{angle } RFT_1,$$

$$\text{and } \therefore \text{each of them} = \frac{1}{2} \text{ angle } TFT_1 = \text{angle } T_1RS,$$

which proves the property above referred to.

*Second Solution.*

The problem may also be solved by dropping perpendiculars from the focus on the given tangents, their points of intersection determining the tangent at the vertex.

PROBLEM 43. (Fig. 39.) *To draw a parabola, the directrix  $CC_1$  and two points  $A$  and  $B$  on the curve being given.*

This is merely the converse of Prob. 40. With the given points as centres and with radii equal to the distance of each from the given directrix describe arcs intersecting in  $F$  and  $F_1$ , either of which may be taken as the focus.

PROBLEM 44. (Fig. 40.) *To draw a parabola, the directrix  $MX$ , a point  $A$  on the curve and a tangent  $YT$  being given.*

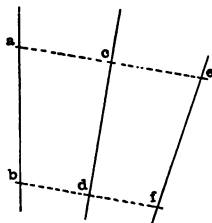
With centre  $A$  describe a circle  $MF M_1$  touching  $MX$ . This will of course be a locus of the focus. At  $S$ , the point of intersection of the given tangent and directrix, construct an angle  $TSF$  equal to the angle between  $MS$  and  $TS$  produced, i.e. = the angle  $MSK$ .  $SF$  will be another locus, i.e. the focus will be at  $F$ , the intersection of the line and circle. The line  $SF$  will evidently meet the circle again beyond  $A$  and this point of intersection will be the focus of a second parabola fulfilling the given conditions, the point of contact of tangent and the point  $A$  being on the same side of the axis.

*Proof.* That the circle is a locus of the focus needs no demonstration: that the line is a locus of the focus is proved, since it has been shewn (Prob. 36) that  $FS$  is always perpendicular to the line joining  $F$  to the point of contact of the tangent through  $S$ , and that therefore the two triangles  $FST$ ,  $LST$ , where  $TL$  is perpendicular to  $MS$ , are equal and similar in all respects; and that therefore angle  $FST$  = angle  $LST$  = angle  $MSK$ .

**PROBLEM 45.** (Fig. 41.) *To draw a parabola, the directrix  $KX$  and two tangents  $RT$ ,  $RT_1$  being given.*

At  $S$ , the point of intersection of  $RT$  with  $KX$ , construct an angle  $TSF$  equal to the angle  $TSK$ . As in the last problem  $SF$

Fig. 41a.



will be a locus of the focus. Similarly, if  $RT_1$  meet the directrix in  $S_1$ , construct an angle  $T_1S_1F$  equal to the angle  $T_1S_1X$ , and  $S_1F$  will be a second locus, therefore the intersection of these lines determines  $F$ , the focus. In the figure the directrix and the tangent  $RT_1$  do not intersect within any reasonable distance, but the line through their intersection making the same angle with the tangent as the tangent does with the directrix can easily be drawn, as shewn in fig. 41a. Let  $ab$ ,  $cd$  be any two converging lines; from any two points  $(a, b)$  on the one, drop perpendiculars  $ac$ ,  $bd$  on the other and produce them: make  $ce = ca$ ,  $df = db$ , then obviously  $ab$  and  $ef$  will pass through the same point on  $cd$  and will be equally inclined thereto.

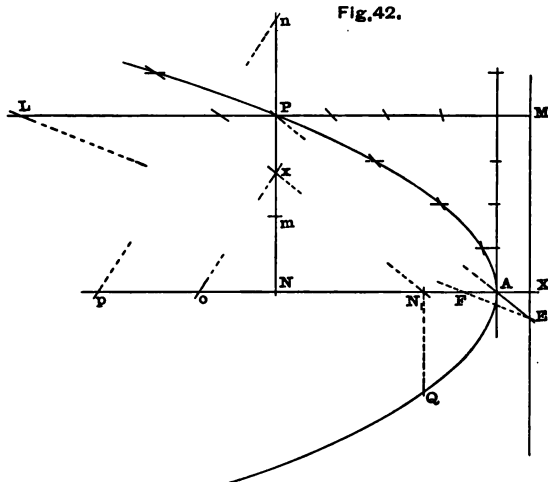
**PROBLEM 46.** (Fig. 42.) *To draw a parabola, the axis  $AN$  and two points  $P$ ,  $Q$  on the curve being given.*

[The two points must not be at equal distances from the axis



whether on the same or on opposite sides of it, nor must they be on the same perpendicular to the axis.]

Draw the ordinates  $PN, QN_1$ , of which let  $PN$  be the greater; the vertex will then obviously lie on the same side of  $N$  as  $N_1$



and beyond it. On  $NP$  produced make  $Pn$  equal to  $QN_1$ , and on  $PN$  make  $Pm$  also equal to  $QN_1$ . Then  $Nm$  is evidently equal to  $PN - QN_1$ . On the axis make  $No = Nm$  and on the same side of  $N$  make  $Np = NP$ . Through  $o$  draw  $ox$  parallel to  $pn$  meeting  $PN$  in  $x$ , and through  $P$  draw  $PA$  parallel to  $N_1x$  meeting the axis in  $A$ .  $A$  will be the vertex of the required parabola and the problem is reduced to Prob. 37.

*Proof.* It is a well-known property in the parabola that  $|PN|^2 = 4AF \cdot AN$  where  $F$  is the focus,  $PN$  an ordinate and  $A$  the vertex.

$$\therefore \overline{PN}^2 : \overline{QN}^2 :: AN : AN_1,$$

or  $\overline{PN}^2 - \overline{QN}^2 : \overline{PN}^2 :: AN - AN_1 : AN,$

i.e.  $(PN + QN_1)(PN - QN_1) : \overline{PN}^2 :: NN_1 : AN,$

$$\text{or } \frac{AN}{NN_1} = \frac{PN}{(PN + QN_1)(PN - QN_1)} PN.$$

If a fourth proportional be taken to

$$PN, PN + QN_1, \text{ and } PN - QN_1,$$

i.e. if a length  $l$  be determined such that

$$PN : PN + QN_1 :: PN - QN_1 : l,$$

the above equation may be written

$$\frac{AN}{NN_1} = \frac{PN}{l},$$

i.e.  $AN$  is a fourth proportional to such length  $l$ ,  $NN_1$  and  $PN$ .

But this is really what has been done, for

$$Np : Nn :: No : Nx,$$

$$\text{i.e. } PN : PN + QN_1 :: PN - QN_1 : Nx,$$

$$\text{i.e. } Nx \text{ is the required length } l,$$

$$\text{and } Nx : NN_1 :: PN : AN.$$

That  $\overline{PN}^2 = 4AF \cdot AN$  may be shewn thus: Join  $PA$  and let it meet the directrix in  $E$ . Join  $EF$  ( $F$  being the focus) and produce it to meet the diameter through  $P$  in  $L$ , while the diameter meets the directrix in  $M$ . Then since  $FA = AX$ ,  $PL = PM = PF$ , for  $ML$  is parallel to  $PX$ , therefore the circle on  $ML$  as diameter goes through  $F$ , and therefore the angles  $MFL$ ,  $MFE$  are both right angles and

$$EX \cdot XM = \overline{FX}^2 = 4AF^2,$$

$$\text{also } AN : AX :: PN : EX \text{ by similar triangles,}$$

$$:: \overline{PN}^2 : EX \cdot MX$$

$$:: \overline{PN}^2 : 4\overline{AF}^2,$$

$$\therefore \overline{PN}^2 = 4AF \cdot AN, \text{ since } AF = AX.$$

**PROBLEM 47.** (Fig. 43.) *To draw a parabola, the axis  $AN$ , a point  $P$  on the curve and a tangent  $OT$  being given.*

[The tangent must not be parallel to the axis, and the point must lie within the angle formed by the tangent and a symmetrical line on the other side of the axis.]

Draw the ordinate  $PN$  and let it meet the given tangent in  $O$ . Make  $NP_1$  on the other side of the axis equal to  $PN$ , and  $P_1$  will



$$\begin{aligned}
 \text{Then } ab \cdot ac &= \overline{av}^2 - \overline{bv}^2 \quad (\text{Euc. II. 6}) \\
 &= 4Fq \cdot (qu - qv) \quad (\text{p. 71}) \text{ if } F \text{ is the focus,} \\
 &= 4Fq \cdot ad,
 \end{aligned}$$

$$\text{and similarly} \quad ab' \cdot aP = 4Fq_1 \cdot ad,$$

$$\therefore ab \cdot ac : ab' \cdot aP :: Fq : Fq_1,$$

i.e. the ratio of the rectangles depends only on the positions of  $q$  and  $q_1$ , and is independent of the position of the point  $a$ .

If the lines  $abc$ ,  $ab'P$  move parallel to themselves until they become the tangents at  $q$  and  $q_1$ , we shall then obtain, if these tangents intersect in  $t$ ,

$$\overline{tq}^2 : \overline{tq_1}^2 :: Fq : Fq_1,$$

$$\text{and} \quad \therefore ab \cdot ac : ab' \cdot aP :: \overline{tq}^2 : \overline{tq_1}^2,$$

but the tangent  $aQ$  may be regarded as a chord cutting the parabola in two coincident points, and therefore if the tangent at  $q$  meet  $aQ$  in  $t$  and  $q_1Q$  meet it in  $m$

$$\begin{aligned}
 ab \cdot ac : aQ^2 &:: \overline{qt}^2 : \overline{tq}^2 \\
 &:: \overline{qt}^2 : \overline{tm}^2.
 \end{aligned}$$

Also if  $Qk$  is the diameter at  $Q$  meeting  $ac$  in  $k$ , by similar triangles

$$\begin{aligned}
 qt : tm &:: ak : aQ, \\
 \therefore ab \cdot ac : aQ^2 &:: \overline{ak}^2 : \overline{aQ}^2,
 \end{aligned}$$

$$\text{or} \quad ab \cdot ac = \overline{ak}^2,$$

which justifies the construction.

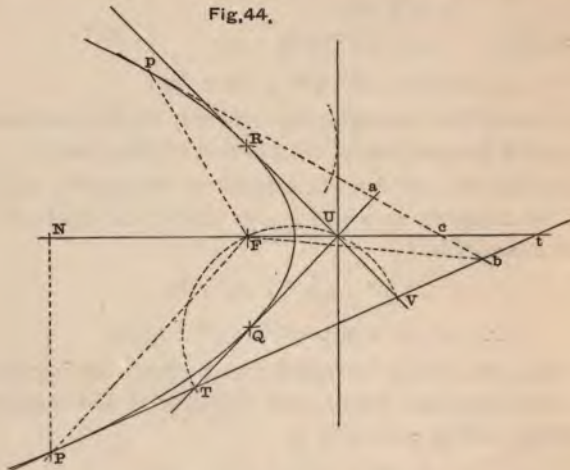
**PROBLEM 48.** (Fig. 44.) *To draw a parabola, the axis  $UN$  and two tangents  $PT$ ,  $QT$  being given.*

[The point  $T$  must not be on the axis.]

If from the point  $U$  in which either of the tangents (as  $QT$ ) cuts the axis, a line  $UR$  be drawn making the same angle with the axis as  $QT$  but on the opposite side of it, this will, by symmetry, be a third tangent to the curve. Let it meet the other tangent ( $PT$ ) in  $V$ . Describe a circle through the three points  $T$ ,  $U$ ,  $V$  (Prob. 20), cutting the axis in  $F$ .  $F$  will be the focus of

the required parabola, and  $FU$  will be the distance from  $F$  of the point of contact of either of the tangents  $QU$ ,  $RU$ . With

Fig. 44.



centre  $F$  and radius  $FU$  describe an arc cutting  $UR$  in  $R$ , with centre  $R$  and the same radius describe a circle, and the directrix will touch this circle and is of course perpendicular to the axis. The problem is therefore reduced to Prob. 36.

*Proof.* The fact that the circle through the points of intersection of three tangents is a locus of the focus is generally true, and is not confined to the case of two tangents meeting on the axis. For draw any tangent  $pab$  meeting the parabola in  $p$ , the two given tangents in  $a$  and  $b$  and the axis in  $c$ , and let  $Tb$  meet the axis in  $t$ . It has been shewn (Prob. 42) that the angle  $abt$  is equal to either of the angles  $pFb$ ,  $PFb$ , also the angle  $Fpc$  = the angle  $Fcp$  = the angle  $bct$ ,

∴ the remaining angle  $Fba$  of the triangle  $Fpb$ ,  
 =       "       "        $btc$        "        $bct$ .

i. e. if two tangents intersect in  $b$  the angle which either makes with  $Fb$  is equal to the angle which the other makes with the axis.

Similarly, since  $QT$ ,  $PT$  intersect in  $T$ , the angle  $F'Ta$  is equal to the angle  $F'tb$ , i.e.  $b'tc$ ,

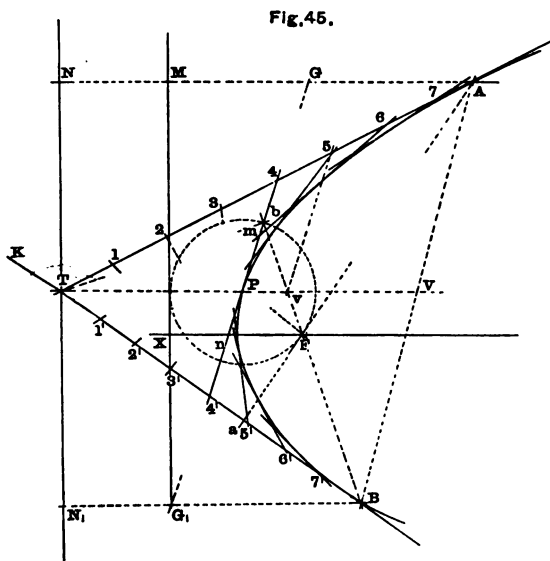
$$\therefore \text{angle } F'ba = \text{angle } F'Ta,$$

or a circle goes round  $aF'Tb$ .

(Euc. III. 27.)

**PROBLEM 49.** (Fig. 45.) *To draw a parabola, two tangents  $AT$ ,  $BT$ , and their points of contact  $A$  and  $B$  being given.*

**First method.** Divide  $AT$ ,  $BT$  into any (the same) number of equal parts; the lines joining opposite points on the two tan-



gents, (i.e. supposing each divided into 8 parts, the lines joining 1 on  $AT$  to  $7'$  on  $BT$ , 2 on  $AT$  to  $6'$  on  $BT$ , and so on,) will be tangents to the curve, which can easily be drawn to touch them all. Or *points* on the curve may be found successively thus. Bisect  $AT$ ,  $BT$  in the points 4,  $4'$ . The line joining these points is a tangent to the curve at its centre point, i.e. bisect 4,  $4'$  in  $P$  and  $P$  is a point on the curve. Similarly the line joining the point of bisection (6) of  $4A$  and the point of bisection of  $(m)$   $4P$

will be a tangent to the curve at *its* centre point, and the line joining the point of bisection (6) of  $4'B$  and the point of bisection ( $n$ ) of  $4'P$  will be a tangent to the curve at *its* centre point, and the method of bisecting the tangents successively may be continued. It is obvious that the point  $m$  found by bisecting  $4P$  is identical with the point of intersection of the line  $44'$  and the line joining 6 on  $A$  to  $2'$  on  $B$ . The focus may be found as the intersection of the circle circumscribing the triangle formed by any three tangents with that circumscribing the triangle formed by any other three, as e.g. the triangles  $4T4'$  and  $5T3'$ , and the directrix may then be determined by Prob. 40.

Second method. The focus may be determined independently, without drawing additional tangents, thus. Join  $AB$ , bisect it in  $V$  and join  $VT$ .  $VT$  will be a *diameter* of the curve, and the curve will pass through  $P$  the centre point of  $VT$ . Bisect  $VT$  in  $P$ . Find a third proportional to  $VT$ ,  $AV$  (Prob. 5), the length of which will be equal to  $2FP$  if  $F$  is the focus. [This may conveniently be done by making  $Tv$  on  $TV$  equal to  $AV$  and drawing a line through  $v$  parallel to  $AV$  to meet  $AT$ . The length ( $l$ ) of this line will be the required third proportional, since  $TV : VA :: Tv$  or  $VA : l$ .]

Describe a circle with centre  $P$  and radius equal to  $\frac{1}{2}l$  which will be a locus of the focus, and the directrix will be a tangent  $MX$  to this circle perpendicular to the diameter  $TV$ . Then  $F$  may be determined as the intersection of a circle, with centre  $A$  and radius  $AM$ , the distance of  $A$  from the directrix, and the previously drawn circle.

Third method. It has been shewn (Prob. 41) that *the exterior angle between any two tangents is equal to the angle which either subtends at the focus*. Therefore if on  $AT$  as chord a segment of a circle be described on the side towards  $B$ , containing an angle  $AFT$  equal to the angle  $ATK$  (Prob. 30), where  $K$  is on  $BT$  produced, this segment will be a locus of the focus. Similarly if a segment containing the same angle be described on  $BT$  towards  $A$ , it will be a second locus and the focus will be at the inter-

section of the two, and the directrix may be determined by Prob. 40.

*Proof.* That the line joining the intersection of tangents to a parabola to the point of bisection of the chord joining their points of contact, is a diameter may be shewn thus. Let  $AB$  be two points on a parabola,  $AT$ ,  $BT$  tangents at the points,  $F$  the focus and  $AN$ ,  $BN_1$  perpendiculars on the directrix meeting  $NTN_1$  parallel to the directrix in  $N$  and  $N_1$ . Join  $FA$ ,  $FB$  and draw  $Ta$  perpendicular to  $FA$  and  $Tb$  perpendicular to  $FB$ . Then the angle  $TAA = \text{angle } TAN$ ,

$$\therefore TN = Ta, \text{ and similarly } TN_1 = Tb.$$

But  $Ta = Tb$ , since it has been shewn that angle  $TFA = \text{angle } TFB$ . (Prob. 42.)

$$\therefore TN = TN_1.$$

If  $TV$  be drawn parallel to  $AN$  or  $BN_1$ , i.e. to the axis, meeting  $AB$  in  $V$ , it will make  $AV : VB :: TN : TN_1$ , i.e.  $AV = VB_1$ , or the diameter through  $T$  bisects  $AB$ . Since  $TN = TN_1$  it follows that any straight line through  $T$  terminated by the diameters  $A$  and  $B$  is bisected in  $T$  and more generally that *every line through the point of intersection of two tangents terminated by diameters through the extremities of the corresponding chord of contact, is bisected by such point of intersection.*

That  $P$ , the point in which the curve meets  $TV$ , bisects  $TV$  and that the tangent at  $P$  is parallel to  $AB$  may be shewn thus:— Since  $AN$ ,  $TV$  and  $BN_1$  are parallel lines, it follows that every line meeting the three is bisected by  $TV$ ; and therefore if the tangent at the point  $P$  be drawn meeting  $AN$  in  $G$  and  $BN_1$  in  $G_1$ ,  $PG = PG_1$ ; but if it meets  $AT$ ,  $BT$  in  $4$  and  $4'$ , it follows as above that  $P4 = 4G$ ,  $P4' = 4'G_1$ , and therefore  $G4$ ,  $4P$ ,  $P4'$  and  $4'G_1$  are all equal, which is only possible if  $GPG_1$  bisects  $TV$  and is parallel to  $AB$ .

$$\text{Hence } T4 = 4A, \quad T4' = 4'B \text{ and } 44' = \frac{1}{2}AB.$$

To shew that  $AV$  is a mean proportional between  $VT$  (or  $2PV$ ) and  $2FP$ , draw  $FU$  parallel to  $AB$  or to  $44'$ , meeting  $PV$  in  $U$ ,



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then the angle  $FUT = \text{angle } 4PU$ ,

$$= \text{angle } F4T, \quad (\text{Prob. 48}),$$

and therefore the circle which it is known can be drawn (Prob. 48) through  $F_4T_4'$ , will pass through  $U$ .

Hence,  $AV$  being twice  $P_4$ ,

$$\overline{AV}^2 = 4\overline{PQ}^2 = 4PU \cdot PT. \quad (\text{Euc. III. 35.})$$

But the angle  $UFP = \text{angle } FP4'$ , since  $FU$  is parallel to  $P4'$ ,

$$= \text{angle } 4'PT = \text{angle } FUP,$$

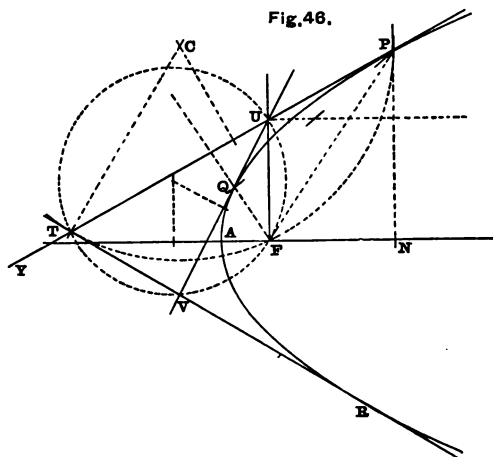
and therefore  $FP = PU$ ; also  $PT = PV$ ,

therefore  $|\overline{AV}|^2 = 4FP \cdot PV$ .

**DEFINITION.** A chord through the focus parallel to the tangent at  $P$  is called *the parameter* of the diameter through  $P$ , and it follows from the above that its length is always equal to  $4FP$ . (See definition of latus-rectum, p. 60.)

**PROBLEM 50.** (Fig. 46.) *To draw a parabola, three tangents,  $TU$ ,  $TV$ ,  $UV$  and the point of contact  $P$  of one of them  $TU$  being given.*

Describe a circle through  $TUV$  (Prob. 20), then  $F$  the focus



lies on this circle (Prob. 48). On  $PT$  describe a segment of a circle containing an angle equal to the exterior angle between the tangents meeting in  $T$ , i.e. the angle  $VTY$ . (Prob. 30.) This segment will be a second locus of the focus (Prob. 42), which will therefore be at the intersection of the segment with the previously drawn circle.

If  $P$  (as in the figure) lies beyond  $U$  the segment must be described on the side of  $TP$  towards  $V$ : but if  $P$  lies between  $T$  and  $U$ , the segment must be described on the other side of  $TP$ , since the focus can never lie inside the triangle  $TUV$  and the angle it contains must be the angle  $UTV$ , since that would then be the exterior angle between the tangents.

[The centre for the segment may conveniently be found by drawing  $TC$  perpendicular to  $VT$  to meet the perpendicular bisector of  $PT$  in  $C$ .]

Construct the angle  $UFQ$  equal to the angle  $PFU$ .  $Q$  will be the point of contact of  $UV$ , and the direction of the axis is determined since it is parallel to the diameter joining  $U$  to the centre point of  $PQ$ . (Prob. 49.) It can then of course be drawn through  $F$ .

Lastly, the vertex may be found since it is the centre point between  $N$  the foot of the ordinate from  $P$  and the point in which  $PT$  cuts the axis (p. 60.)

The point of contact  $R$ , of  $TV$ , may of course be determined without drawing the curve by making the angle  $TFR$ =angle  $TFP$ .

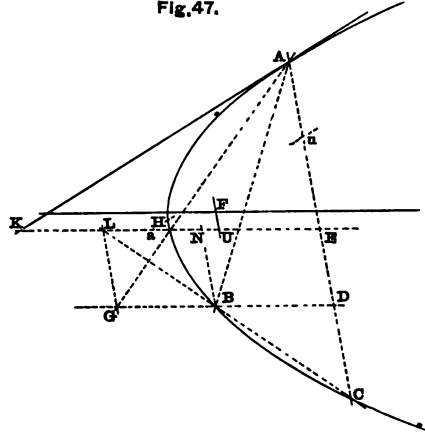
The construction is evident from preceding problems.

**PROBLEM 51.** (Fig. 47.) *To draw a parabola, three points  $A, B, C$ , on the curve, and the direction of the axis, as  $BD$ , being given.*

Draw lines through  $AB, BC, CA$ , and let  $BD$  parallel to the given direction of axis meet  $AC$  in  $D$ . Bisect  $AC$  in  $E$  and draw  $EL$  parallel to  $BD$  to meet  $BC$  in  $L$ . Draw  $LG$  parallel to  $AC$  to meet  $BD$  in  $G$ . Join  $AG$  and it will cut  $EL$  in  $H$ , the vertex of the diameter through  $E$ .

If  $HK = HE$ ,  $AK$  will be the tangent at  $A$  and the focus may be found by taking  $HU$  such that  $\overline{AE}^2 = 4HE \cdot HU$ , i.e. taking

Fig. 47.



$2HU$  a third proportional to  $2HE$ ,  $AE$ ; drawing  $UF$  parallel to  $AC$  and making  $HF = HU$ . [If we take a third proportional to  $EK$ ,  $AE$  it will be  $2HU$ . This may conveniently be done by making  $Ea = EA$  and drawing  $au$  parallel to  $AK$ .  $Eu$  will be the required third proportional. The problem reduces to Prob. 40.]

*Proof.* To shew that  $H$  is the vertex of the diameter through  $E$ . Draw  $BN$  parallel to  $AC$  meeting  $EH$  in  $N$ .  $BN = ED$ , and  $DA \cdot DC = \overline{AE}^2 - ED^2 = AE^2 - BN^2$  (Euc. II. 5);

but in any parabola

$$AE^2 = 4 \cdot FH \cdot HE,$$

and

$$BN^2 = 4 \cdot FH \cdot HN,$$

$$\therefore AE^2 - BN^2 = 4FH \cdot EN = 4FH \cdot BD,$$

$$\therefore DA \cdot DC : AE^2 :: BD : HE;$$

but in the figure

$$BD : EL :: DC : CE \text{ or } EL = \frac{BD \cdot AE}{DC},$$

and  $AE : EH :: AD : DG,$   
 $:: AD : EL,$   
 $:: AD : \frac{BD \cdot AE}{DC},$

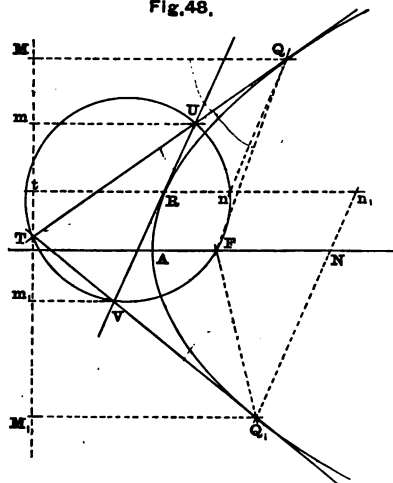
i.e.  $DA \cdot DC : AE^2 :: BD : HE,$

or  $HE$  has been determined of the proper length.

**PROBLEM 52.** (Fig. 48.) *To draw a parabola, three tangents  $UT, TV, VU$  and the direction of the axis, as  $AN$ , being given.*

Through  $T$  draw  $MTM_1$  perpendicular to the given direction of the axis. It is a known property of the parabola that if the

Fig. 48.



portion of any tangent  $UV$  intercepted between two others  $UT, TV$  be projected on any line parallel to the directrix as on  $MM_1$  by lines  $Um, Vm_1$  perpendicular to  $MM_1$ , then any other tangent to the curve between the points of contact of  $TU, TV$  will have the same projected length  $mm_1$  on the axis. If therefore  $TM, TM_1$  be each made equal to  $mm_1$ , lines through  $M$  and  $M_1$  perpendicular to  $MM_1$  will intersect  $TU, TV$  respectively in  $Q$  and  $Q_1$ ,

the points of contact of  $TU$  and  $TV$ . The problem is therefore reduced to Prob. 49, or it may be completed by utilising other known properties of the curve already demonstrated, e.g.—making the angle  $TQF$  equal to angle  $TQM$ ,  $QF$  is a locus of the focus; similarly  $Q_1F$  (the angle  $TQ_1F$  being made equal to angle  $TQ_1M_1$ ) is a second locus, and  $F$ , the focus, is therefore the intersection of  $QF$ ,  $Q_1F$ .

Again, the circle round  $UT$ ,  $TV$ ,  $VU$  is known to be a locus of the focus (Prob. 48), and the angle  $UFQ$  is known to be equal to the angle  $TUV$ . Prob. 42. Therefore, if on  $UQ$  a segment of a circle be described containing an angle equal to the angle  $TUV$  (Prob. 30), the intersection of this segment with the above circle will determine  $F$ . Any number of tangents to the curve between  $Q$  and  $Q_1$  can be at once drawn without previously determining the focus by measuring the length  $mm_1$  anywhere on  $MM_1$  between  $M$  and  $M_1$  and from the extremities drawing perpendiculars to  $MM_1$  to meet  $TQ$ ,  $TQ_1$ . Any pair of such points being joined will of course give a tangent to the curve.

*Proof.* That the projected length on  $MM_1$  of the portion of any tangent intercepted between  $TQ$ ,  $TQ_1$  is constant may be shewn thus. Let  $R$  be the point of contact of  $UV$  and let the diameter through  $R$  meet  $MM_1$  in  $t$ . Draw  $Qn$ ,  $Q_1n_1$  parallel to  $UV$  meeting  $tR$  in  $n$  and  $n_1$ . Then  $UR = \frac{1}{2} Qn$  (Prob. 49) and therefore  $tm = \frac{1}{2} tM$ . Similarly  $tm_1 = \frac{1}{2} tM_1$ .

Therefore  $mm_1 = \frac{1}{2} MM_1 = \text{constant}$ , since  $MM_1$  is the projection of the chord of contact of two fixed tangents.

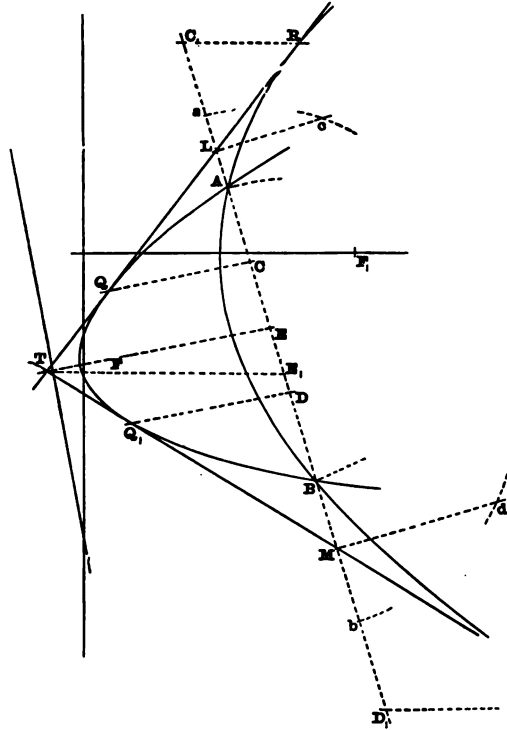
PROBLEM 53. (Fig. 49.) *To draw a parabola, two points  $A$ ,  $B$  on the curve and two tangents  $TL$ ,  $TM$  being given.*

[The tangents must not be parallel and the points must not be on opposite sides of either tangent.]

Draw a line through  $A$  and  $B$  meeting the given tangents in  $L$  and  $M$ . Take  $LC$  on  $LM$  a mean proportional between  $LA$  and  $LB$  (Prob. 5), and  $MD$  on  $ML$  a mean proportional between  $MB$  and  $MA$ . Bisect  $CD$  in  $E$ .  $TE$  will be the direction of the axis

of a parabola fulfilling the required conditions and  $CQ$ ,  $DQ_1$  drawn parallel to  $TE$  to meet the given tangents will determine  $Q$  and  $Q_1$ ,

Fig. 49.



their points of contact. The problem therefore reduces to Prob. 49, or may be completed similarly to the preceding. Since  $LC$  and  $MD$  may be set off on either side of  $L$  and  $M$ , as  $LC_1$ ,  $MD_1$  in the figure, the point of bisection  $E_1$  of  $C_1D_1$  determines  $TE_1$  the direction of the axis of a second parabola fulfilling the required conditions. Further, either  $C_1D$  or  $CD_1$  may also be taken as the segment to be bisected, and there are consequently *four* solutions.

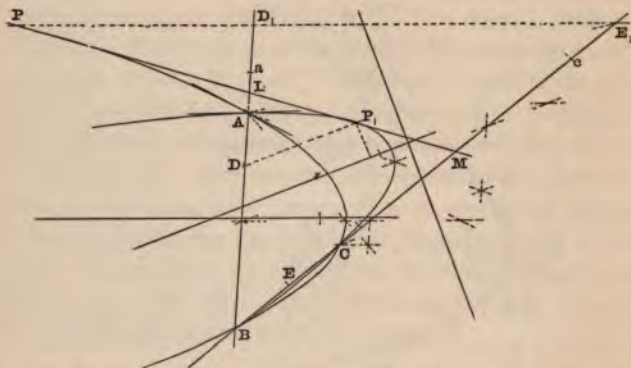
The proof depends entirely on the property of the parabola already referred to in Prob. 47.

PROBLEM 54. (Fig. 50.) *To draw a parabola, three points  $A, B, C$ , and a tangent  $LM$  being given.*

[The points must all be on the same side of the tangent.]

Join two pairs of the given points as  $AB, BC$  and let the joining lines cut the given tangent in  $L$  and  $M$ . On  $LB$  take  $LD$  a mean

Fig. 50.



proportional between  $LA$  and  $LB$  (Prob. 5), and on  $MB$  take  $ME$  a mean proportional between  $MC$  and  $MB$ . Then by the property of the parabola already referred to (Prob. 47) a line through  $D$  parallel to the axis of a parabola through  $A$  and  $B$  and touching  $LM$ , will pass through the point of contact of  $LM$  with such parabola; and a line through  $E$  parallel to the axis of a parabola through  $B$  and  $C$  and touching  $LM$  will pass through the point of contact of  $LM$  with such parabola. Hence the line joining  $DE$  will be parallel to the axis of a parabola which can be described through  $AB$  and  $C$  to touch the given line, and its intersection with  $LM$  will determine the point of contact of such parabola.

Since  $LD, ME$  can be set off on either side of  $L$  and  $M$  (as  $LD_1, ME_1$ ), similarly the line joining  $D_1$  and  $E_1$  will be parallel to the axis of a second parabola fulfilling the conditions of the problem; its point of contact being  $P_1$ : and similarly  $DE_1$  and  $D_1E$  will determine the direction of the axes of two more such parabolas. The line  $DE_1$  determines  $P_1$  as the point of contact.

Hence there are four solutions, and the problem in either case is reduced to Prob. 51. In the fig. two of the four parabolas are drawn, viz. those whose axes are parallel to  $D_1E_1$  and  $DE_1$  respectively; the necessary construction in each case being indicated.

It might be considered at first sight that if a mean proportional were taken between the segments  $NA$ ,  $NC$  of the line joining  $AC$ , the third pair of the given points, cutting the given tangent in  $N$ , two additional points would be obtained which, being joined to either  $D$ ,  $D_1$ ,  $E$  or  $E_1$ , would give the directions of axes of additional parabolas. This however is not so, since it will be found that the points thus obtained coincide with the intersections of  $ED$ ,  $E_1D_1$ , and of  $DE_1$ ,  $ED_1$ , respectively, and therefore no more solutions than the four already mentioned are obtainable.

PROBLEM 55. (Fig. 51.) *To draw a parabola, a point  $A$  on the curve and three tangents  $BC$ ,  $CD$ ,  $DB$  being given.*

[No two of the tangents must be parallel, and the given point must not lie within the triangle formed by the tangents, nor so that any one tangent lies between it and either of the remaining tangents.]

Let  $C$  be the vertex of the triangle formed by the tangents, which cannot be reached from the given point without crossing  $BD$ . Through  $B$  draw  $BE$  parallel to  $CD$  and through  $D$  draw  $DE$  parallel to  $CB$ , meeting  $BE$  in  $E$ . Through  $C$  draw  $CK$  parallel to  $BD$  and join  $EA$  meeting  $CK$  in  $K$ ,  $CB$  in  $L$ , and  $BD$  in  $M$ .

First let  $A$  lie between  $E$  and  $K$ ; complete the harmonic range  $KA EA_1$ , i.e. find a point  $A_1$  beyond  $E$  on  $KL$  such that

$$KA : KA_1 :: AE : EA_1 \quad (\text{Prob. 12.})$$

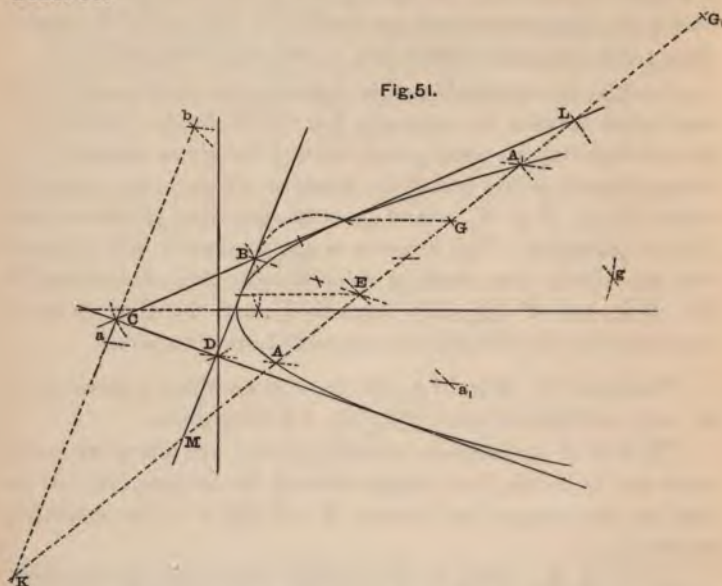
[Through  $A$ ,  $K$  draw  $Aa$ ,  $Ka$  any two lines intersecting in  $a$ , produce  $aA$  to  $a_1$  making  $Aa_1 = Aa$ . Join  $a_1E$  and produce it to meet  $Ka$  in  $b$ . Draw  $ba_1$  parallel to  $aA$  and it will intersect  $KA$  in the required point.]

Then  $A_1$  will be a point on the curve and the problem reduces to Prob. 53.

Second, let the given point lie beyond  $E$  as  $A_1$ , then, completing the harmonic range  $A_1 EA K$  (Prob. 11),  $A$  will be a



second point on the curve and the problem again reduces to Prob. 53.



In completing the figure, one of the tangents employed should be the one situated as  $BD$  in the figure, because it is necessary to take a mean proportional between the segments of the chord  $AA_1$ , included between the tangent and the curve, i.e. to take a mean proportional between  $MA$  and  $MA_1$ ; but it will be found that  $ME$ ,  $MK$  are each equal to such mean proportional, and therefore  $E$  and  $K$  can be at once used without any further construction. If  $CB$  is the second tangent made use of, a mean proportional  $LG$  or  $LG_1$  must be determined between  $LA$ ,  $LA_1$  (Prob. 5), and two of the four parabolas which can be constructed by means of pairs of the points  $K$ ,  $E$ ,  $G$ ,  $G_1$  to pass through  $A$  and  $A_1$  and to touch  $BL$ ,  $BD$  will also touch  $CD$ . There is an ambiguity as to which particular pairs of points must be selected, but this can easily be settled by trial in any given case. In the fig. it will be found that the pairs  $E$ ,  $G$  and  $E$ ,  $G_1$  are those required, and that

the pairs  $K, G$  and  $K, G_1$  give parabolas which while touching  $BC, DB$ , do not touch  $CD$ .

There are in general *two* solutions.

*Proof.* It is shewn at the end of Chap. iv. among the harmonic properties of conics, that the three diagonals of a complete quadrilateral circumscribing a conic form a self-conjugate triangle. It is easily proved analytically that every parabola touches the line at infinity, i.e. has one tangent situated at an altogether infinite distance. Now  $BE$  and  $DE$  meet  $CD, CB$  respectively in infinitely distant points, pass, that is, through the points in which this infinitely distant tangent meets  $CB$  and  $CD$ , they are therefore diagonals of the circumscribing quadrilateral formed by the three given and the infinitely distant tangent, and its third diagonal must be the line  $CK$  since this meets  $BD$  in infinitely distant points.  $E$  is therefore the pole of the line  $CK$ , and conversely the polar of  $K$  passes through  $E$ .

But a straight line drawn through any point is divided harmonically by the point, the curve and the polar of the point (see end of Chap. iv.), therefore  $A_1$  must be a point on the curve.

**PROBLEM 56.** (Fig. 52.) *To draw a parabola to pass through four given points  $A, B, C, D$ .*

[The points must not lie at the angles of a parallelogram, and must be so situated, that being joined in pairs, the two points of each pair are both on the same side, or on opposite sides of the point of intersection of the joining line\*.]

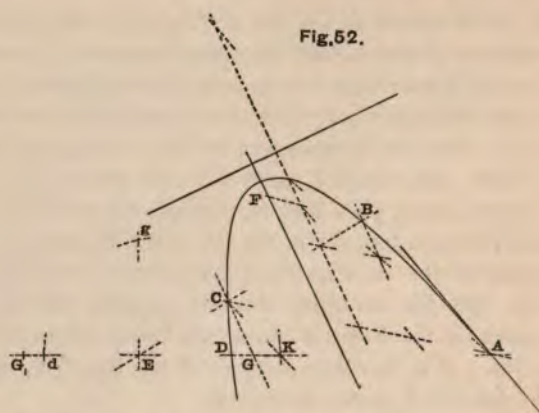
Join  $BC, AD$  to meet in  $E$ . Through  $C$  draw  $CK$  parallel to  $AB$  meeting  $AD$  in  $K$ . Take a mean proportional  $EG$  between  $ED$  and  $EK$  (Prob. 5) and  $CG$  will be the direction of the axis of the required parabola. The Problem is therefore reduced to Prob. 51.

Since the distance  $EG$  may be set off on either side of  $E$  as  $EG_1$ , the line  $CG_1$  will be the direction of the axis of a second parabola fulfilling the given conditions.

\* Puckle's *Conic Sections*. Fourth Edition, Art. 313, Ex. 1.

*Proof.* From the construction

$$EB : EA :: EC : EK,$$



and

$$EK : EG :: EG : ED;$$

$$\therefore EB : EA :: EC : \frac{EG^2}{ED},$$

or

$$EG^2 : EC :: ED . EA : EB,$$

which may be written

$$EG^2 : EC^2 :: ED . EA : EC . EB,$$

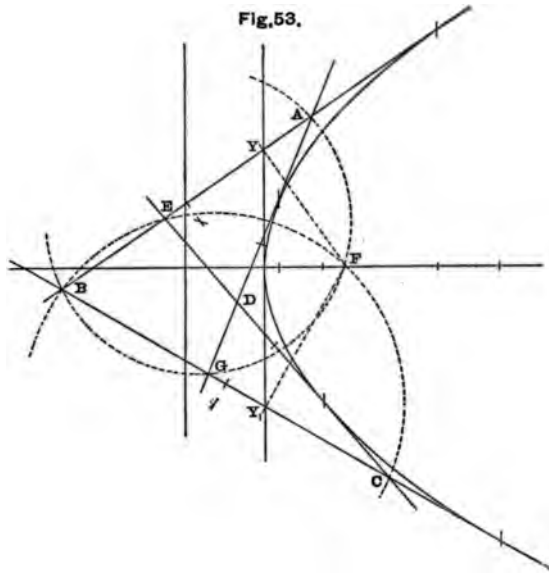
a relation which is known to hold in the parabola. (Besant's *Geom. Conics*, 3rd Ed., Art. 213.)

PROBLEM 57. (Fig. 53.) *To draw a parabola to touch four given lines AB, BC, CD, DA, no two of which are parallel.*

Let CD meet AB in E and AD meet BC in G.

The circle circumscribing the triangle formed by any three of the lines will be a locus of the focus (Prob. 48), which may therefore be determined as the intersection of the circles circumscribing any two of such triangles. In the figure, the circles circumscribing BCE and ABG are drawn. They intersect in F,

the focus. The tangent at the vertex can be at once determined, by dropping perpendiculars from  $F$  on any two of the given



tangents as  $FY$ ,  $FY_1$  perpendiculars on  $AB$ ,  $BC$ ;  $Y$  and  $Y_1$  are points on the tangent at the vertex. (Notes to Problem 36.)

**PROBLEM 58.** (Fig. 54.) *To determine the centre of curvature at any point  $P$  of a given parabola.*

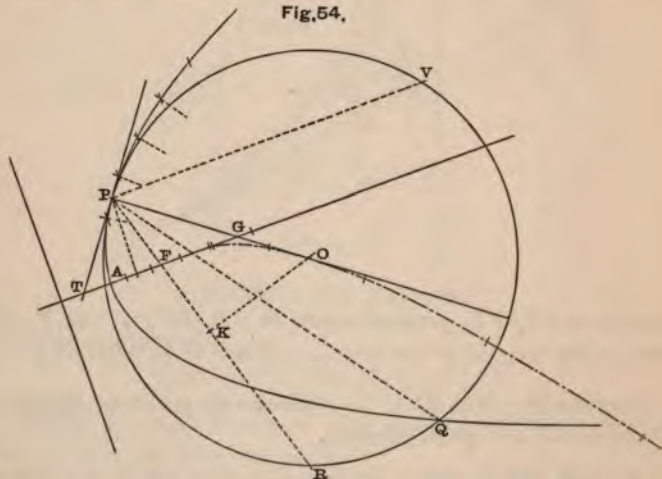
[A circle can be drawn through any three points of a curve, but cannot in general be drawn through a greater number taken arbitrarily. If a circle be drawn through three points of a curve and the outside points be conceived to gradually move up to the centre one, the circle in the limiting position it assumes when the points approach indefinitely near to each other so as ultimately to coincide, is called the *circle of curvature* at the point, and its centre is called the *centre of curvature*. The circle is said to pass through three consecutive points of the curve, and obviously has closer contact with it at the point than any other circle can

have, since it is not possible to draw a circle through *four* consecutive points. The centre of curvature will necessarily lie on the normal at the given point, and any circle having its centre on the normal and passing through the point really passes through two consecutive points of the curve, since curve and circle have a common tangent.]

$F$  is the focus,  $PT$  the tangent, and  $PG$  the normal at the point  $P$  of the given parabola.

Join  $PF$  and produce it to  $K$ , making  $FK$  equal to  $FP$ . Draw  $KO$  perpendicular to  $PK$  to intersect the normal at  $P$  in  $O$ .  $O$  will be the *centre of curvature* at  $P$ .

Fig. 54.



If the circle of curvature cuts the parabola again in  $Q$ , it will be found that  $PQ$ , the common chord, makes the same angle with the axis as  $PT$ , the tangent, does, and that

$$PQ = 4PT.$$

The focal chord  $FR$  of the circle of curvature is known to be in length equal to  $4FP$ , and it is on this known value of the focal chord that the construction depends.

The chord  $(PV)$  of the circle of curvature through  $P$  parallel

to the axis is also equal to  $4FP$ , since this chord and  $PR$  are equally inclined to the tangent at  $P$ .

The length  $PO$  of the radius of curvature may also be determined by taking a fourth proportional to  $FY$ ,  $FP$  and  $2FP$ , where  $FY$  is the perpendicular from  $F$  on the tangent at  $P$ .

The locus of the centre of curvature of any curve is called the *Evolute* of that curve; and the original curve, when considered with respect to its evolute, is called an *Involute*. The chain-dotted curve in Fig. 54 is the evolute of the portion of the parabola lying above the axis.

Normals to the curve are tangents to the evolute; and since the focal radius of curvature at the vertex  $= 2 \cdot AF$ , the evolute must touch the axis at a point  $= 2 \cdot AF$  from  $A$ .

If the ordinate of the point of intersection of the curve and evolute be drawn meeting the axis in  $N$ , it will be found that

$$AN = 8 \cdot AF = \text{twice the latus rectum.}$$

The evolute of the parabola is a curve known as the semi-cubical parabola.

**PROBLEM 59.** *To draw a parabola to touch two given circles, the axis being the line joining the centres.*

Let  $C$  be the centre of the larger circle,  $c$  that of the smaller,  $R$  and  $r$  their radii. Determine a fourth proportional to  $2Cc$ ,  $R+r$ , and  $R-r$ . From  $C$  towards  $c$  set off on  $Cc$  a length  $CN$  equal to this fourth proportional, i.e. a length such that

$$CN : R-r :: R+r : 2Cc.$$

Draw  $NP$  perpendicular to  $Cc$  meeting the circle in  $P$ , and  $P$  will be the required point of contact of the curve. The problem therefore reduces to Prob. 47, the given point being also the point of contact of the given tangent.

## EXAMPLES ON CHAPTER III.

1. Draw a parabola, the focus  $F$ , the position of the axis ( $FT$ ) and a tangent ( $PT$ ) being given.

(From  $F$  draw  $FY$  perpendicular to  $PT$  meeting it in  $Y$ , and from  $Y$  draw  $YA$  perpendicular to  $FT$  meeting it in  $A$ .  $A$  will be the vertex of the required parabola.)

2. Draw a parabola, the focus  $F$ , a tangent  $PT$  and the length of the latus rectum being given.

(With centre  $F$  and radius equal to one-fourth of the given latus rectum, describe a circle; from  $F$  draw  $FY$  perpendicular to the given tangent meeting it in  $Y$ , and from  $Y$  draw tangents to the circle. Either point of contact will be the vertex of the required parabola (two solutions). The given tangent must not cut the circle.)

3. Draw a parabola, two points ( $P$ ,  $Q$ ), the tangent at one of them ( $PT$ ), and the direction of the axis being given.

(Bisect  $PQ$  in  $V$ , draw  $VT$  parallel to given direction of axis meeting the given tangent in  $T$ ;  $QT$  is the tangent at  $Q$ , and problem reduces to Prob. 49.)

4. Draw a parabola, the vertex ( $P$ ) of a diameter, and a corresponding double ordinate  $QQ_1$  being given.

(Bisect  $QQ_1$  in  $V$ .  $PV$  will be a diameter; on  $VP$  produced make  $PT = PV$ .  $TQ$  and  $TQ_1$  are the tangents at  $Q$  and  $Q_1$ , and problem reduces to Prob. 49.)

5. Draw the locus of the foci of the parabolas which have a common vertex ( $A$ ) and a common tangent  $PT$ .

(The parabola which has  $A$  for vertex, the perpendicular on  $PT$  as axis, and the distance of  $PT$  from  $A$  as latus rectum.)

6. Inscribe in a given parabola a triangle having its sides parallel to three given straight lines  $AB$ ,  $BC$ ,  $CA$ .

(Draw  $BD$  parallel to the axis of the parabola meeting  $AC$  in

$D$  and  $CE$  parallel to the axis meeting  $AB$  in  $E$ . Draw a tangent to the parabola parallel to  $DE$  (p. 61) and from  $P$  its point of contact draw  $PQ, PR$  parallel to  $AB, AC$  meeting the parabola again in  $Q, R$ .  $PQR$  will be the required triangle.)

7. Draw a parabola with a given focus, and to touch a given circle at a given point.

[Let  $F$  be the focus,  $P$  the point on the circle, draw  $PT$  the tangent, and construct an angle  $TPM =$  the angle  $FPT$ . The axis of the required parabola will be parallel to  $PM$ .]

8. Shew that if tangents be drawn to a parabola from any point  $O$ , and a circle be described with the focus as centre, passing through  $O$  and cutting the tangents in  $P$  and  $Q$ ,  $PQ$  will be perpendicular to the axis, and its distance from  $O$  is twice its distance from the vertex.

9. Draw a circle to touch a parabola in  $P$ , and to pass through the focus. Let it meet the parabola again in  $Q$  and  $Q_1$ ; draw a focal chord parallel to the tangent at  $P$ , and shew that the circle on this chord as diameter will pass through  $Q, Q_1$ , and that the focal chord and  $QQ_1$  will intersect on the directrix.

10. Draw any right-angled triangle  $DEF$  ( $E$  being the right angle). Describe a parabola with focus  $F$  and to touch  $ED$  at  $D$ , and shew that if any circle be described to pass through  $D$  and  $F$  and cutting  $ED$  produced in  $P$ , the tangent to it at  $P$  will also be a tangent to the parabola.

11. Given two lines  $PR, QR$ , and a point  $P$  on one of them, shew that any point on the circumference of the circle passing through  $P$  and  $R$  and touching  $QR$  may be taken as the focus of a parabola passing through  $P$  and to which the given lines shall be tangents.

12.  $AB$  is the diameter of a circle; with  $A$  as focus and any point on the semi-circumference of which  $A$  is the centre as foot of directrix describe a parabola, and shew that it will touch the diameter perpendicular to  $AB$ .



13. If  $\triangle APC$  be a sector of a circle of which the radius  $CA$  is fixed, and a circle be described touching the radii  $CA$ ,  $CP$  and the arc  $AP$ , shew that the locus of the centre of this circle is a parabola and describe it.

14. Given a segment of a circle, describe the parabola which is the locus of the centres of the circles inscribed in it.

15. If from a point  $P$  of a circle  $PC$  be drawn to the centre, and  $R$  be the middle point of the chord  $PQ$  drawn parallel to a fixed diameter  $ACB$ , describe the locus of the intersection of  $CP$ ,  $AR$ , and shew that it is a parabola.

16. Describe a parabola with latus rectum  $= 2.7$  units, and in it draw a series of parallel chords inclined at  $60^\circ$  to the axis. Shew that the locus of the point which divides each chord into segments containing a constant rectangle  $= 4$  sq. units in area, is a parabola, the axis of which coincides with the axis of the original parabola and with the latus rectum  $= 2.1$  units.

17. Draw a parabola to touch the three sides of a given triangle, one of them at its middle point; and shew that the perpendiculars drawn from the angles of the triangle upon any tangent to the parabola are in harmonical progression.

18. Given two unequal circles (centres  $G$  and  $g$ , radii  $R$  and  $r$ ) touching each other externally, from  $G$  the centre of the larger circle make  $GN$  on  $Gg$  towards  $g = \frac{R-r}{2}$ . Draw  $NP$  perpendicular to  $Gg$  meeting the circle in  $P$  and describe a parabola with  $Gg$  as axis and to touch the circle in  $P$  (Prob. 47), and shew that it will also touch the smaller circle.

19. Given a point  $F$  and two straight lines intersecting in  $O$ ; describe a parabola with  $F$  as focus and to touch the given lines (Prob. 42); and shew that if any circle be described passing through  $O$  and  $F$  and meeting the lines in  $P$  and  $Q$ ,  $PQ$  will be a tangent to the parabola.

20. Draw the parabola which is the locus of the centre of a circle passing through a given point and cutting off a constant intercept on a given straight line.

(The point is the focus and a perpendicular to the line the axis.)

21. Given four tangents to a parabola, shew that the directrix is the radical axis of the system of circles described on the diagonals of the quadrilateral as diameters.

22. Given the focus  $F$ , a point  $P$  on the curve and a point  $L$  on the directrix, describe the parabola.

[Tangents from  $L$  to the circle described with centre  $P$  and radius  $PF$  are the directrices of two parabolas fulfilling required conditions.]

23. Given a focus  $F$ , a tangent  $PT$ , and a point  $L$  on the directrix, describe the parabola.

[From  $F$  draw a perpendicular  $FY$  to  $PT$  meeting it in  $Y$ ; produce  $FY$  to  $f$  and make  $Yf = FY$ :  $f$  is a second point on the directrix.]

24. Given three tangents to a parabola and a point on the directrix, draw the curve.

[The ortho-centre of the triangle formed by the tangents is a second point on the directrix.]

## CHAPTER IV.

### THE ELLIPSE.

THE ellipse has already been defined (p. 56) as the locus of a point which moves in a plane so that its distance from a fixed point in the plane is always in a constant ratio, less than unity, to its distance from a fixed line in the plane. The corresponding definition in the case of the parabola furnishes immediately the best condition for the geometrical construction of that curve, but this is not so with the ellipse. The ellipse can be more easily constructed geometrically from a property which will be shewn immediately to be involved in the above definition, and in virtue of which the curve may be defined as follows:—

DEF. The ellipse is the locus of a fixed point on a line of constant length moving so that its extremities are always on two fixed straight lines perpendicular to each other.

In Fig. 55 let  $ACA_1$ ,  $BCB_1$  be two straight lines intersecting each other at right angles in  $C$ . If a length (as  $ab$ ) be marked off on the smooth edge of a slip of paper, and the slip be moved round so that the point  $a$  is always on the line  $BCB_1$  and the point  $b$  on  $ACA_1$ , then any point as  $P$  on the edge of the paper will trace out an ellipse. When the edge of the slip coincides with  $ACA_1$ , the tracing point will evidently be at a distance  $CA$  from  $C$  equal to  $aP$ , and when it coincides with  $BCB_1$  the tracing point will be at a distance  $CB$  from  $C$  equal to  $bP$ . By this method of construction the curve is evidently symmetrical about both the lines  $ACA_1$  and  $BCB_1$ , i.e. if  $CA_1$  be made equal to  $CA$ ,  $A_1$  will be a point on the curve, and if  $CB_1$  be made equal

to  $CB$ ,  $B_1$  will be a point on the curve. It is moreover obvious that  $ACA_1$  is the longest and  $BCB_1$  the shortest line which can be drawn through  $C$  and terminated by the curve.

DEF. The line  $ACA_1$  is called the *major axis*, the line  $BCB_1$  the *minor axis*, the point  $C$  the *centre*, and the points  $A, A_1$  *vertices* of the curve.

From  $B$ , the extremity of the minor axis, as centre with radius  $CA$  (the semi-major axis), describe arcs cutting the major axis in  $F$  and  $F_1$ ; through  $B$  draw  $BM$  parallel to  $CA$ , from  $F$  draw  $FM$  perpendicular to  $BF$  meeting  $BM$  in  $M$ , and draw  $MX$  perpendicular to  $CA$  meeting it in  $X$ .

$F$  will be the *focus* and  $MX$  the *directrix* (see definition, page 56).

From the similar triangles  $FBM, CFB$ ,

$$FB : BM :: CF : CA :: CA : CX,$$

since

$$FB = CA \text{ and } BM = CX;$$

$$\therefore CF : CA - CF :: CA : CX - CA,$$

i.e.

$$CF : FA :: CA : AX,$$

or

$$FA : AX :: CF : CA :: FB : BM;$$

also since

$$CF : CA :: CA : CX,$$

$$\therefore CA - CF : CF + CA :: CX - CA : CA + CX,$$

or

$$FA : FA_1 :: AX : A_1X,$$

i.e.

$$FA : AX :: FA_1 : A_1X;$$

therefore  $A, B$  and  $A_1$  are points satisfying the original definition.

DEF. A circle described on the major axis as diameter is called the *auxiliary circle*.

Through any point  $P$  on the ellipse draw the ordinate  $PN$  (perpendicular to major axis) meeting the axis in  $N$  and the auxiliary circle in  $Q$ . Since  $QN$  is parallel to  $BC$  and  $CQ = aP$ ,

$$\therefore aP \text{ is parallel to } CQ,$$

$$\therefore PN : QN :: Pb : QC$$

$$:: BC : AC,$$

$$\text{or } \overline{PN}^2 : \overline{QN}^2 :: \overline{BC}^2 : \overline{AC}^2;$$

but it is known that in the circle

$$\overline{QN}^2 = AN \cdot NA_1,$$

$$\therefore \overline{PN}^2 : AN \cdot NA_1 :: \overline{BC}^2 : \overline{AC}^2.$$

This is a very important property of the ellipse and will now be shewn to result from assuming the ratio  $FP : NX$  to be constant.

Through  $P$  draw  $PA$ ,  $PA_1$  meeting the directrix in  $E$  and  $H$ . Join  $FH$  and draw  $PLK$  perpendicular to the directrix meeting  $FH$  in  $L$  and the directrix in  $K$ .

Since  $PK$  is parallel to  $A_1X$ ,

$$\begin{aligned} \therefore PL : PK &:: FA_1 : A_1X \\ &:: FA : AX. \end{aligned}$$

But by supposition  $FP : PK :: FA : AX$ ,  
therefore  $FP = PL$ , and the angle  $LFP = FLP =$  the alternate angle  $LFX$ ;

i.e.  $FL$  bisects the angle  $PFH$ ;

similarly  $FE$  bisects the angle between  $FX$  and  $PF$  produced, therefore the angle  $EFH$  is a right angle, since it is made up of the two angles  $EFX$  and  $HFX$ .

By the similar triangles  $PAN$ ,  $AEX$ ,

$$PN : AN :: EX : AX,$$

also  $PN : A_1N :: HX : A_1X$ ,

$$\begin{aligned} \therefore \overline{PN}^2 : AN \cdot NA_1 &:: EX \cdot HX : AX \cdot A_1X \\ &:: FX^2 : AX \cdot A_1X, \end{aligned}$$

since  $EFH$  is a right angle;

i.e.  $PN^2$  is to  $AN \cdot NA_1$  in a constant ratio.

Hence taking  $PN$  coincident with  $BC$ , in which case

$$AN = NA_1 = AC,$$

$$\overline{BC}^2 : \overline{AC}^2 :: \overline{FX}^2 : AX \cdot A_1X,$$

and  $\therefore PN^2 : AN \cdot NA_1 :: BC^2 : AC^2$ .

This of course shews that the point  $P$  is the same whether determined as the locus of a fixed point on a line of constant length

sliding between two fixed rectangular axes or as the locus of a point which moves so that its distance from a fixed point ( $F$ ) is in a constant ratio to its distance from a fixed line ( $MX$ ), i.e. the two definitions of the ellipse already given are really identical. From the symmetry of the curve it is evident that  $F_1$  is a second focus and  $M_1X_1$  a second directrix.

Five geometrical conditions are generally necessary to determine an ellipse, and the ellipse shares with the hyperbola the property of satisfying five geometrical conditions. One or other of these curves can generally be drawn to pass through five given points or to touch five given straight lines, or to pass through two given points and touch three given lines, or to fulfil any five similar conditions. Which curve will satisfy the given conditions depends of course upon the relative positions of the given points and lines, and the necessary limitations will be noticed in discussing the separate problems. As in the case of the parabola the giving of certain points and lines is really equivalent in each case to the giving of two geometrical conditions; of these may be mentioned the centre, the foci, and the axes.

The eccentricity of the ellipse is (p. 57) the numerical value of the above fixed ratio; it is generally denoted by  $e$  and calling

$$CA = a$$

and

$$CB = b,$$

its value is

$$e = \frac{\sqrt{a^2 - b^2}}{a},$$

as is evident from the similar triangles  $FBM$ ,  $FCB$ .

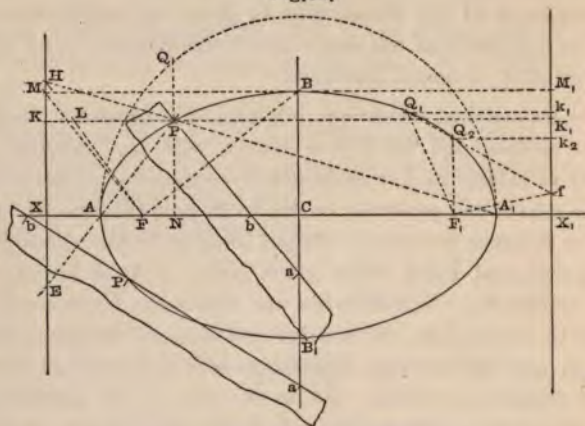
**PROBLEM 60.** (Fig. 55.) *To describe an ellipse having given axes  $AA_1$ ,  $BB_1$ .*

*First Method.* Draw two lines perpendicular to each other intersecting in  $C$ . Set off  $CA$ ,  $CA_1$  each equal to  $\frac{1}{2}AA_1$ , and  $CB$ ,  $CB_1$  each equal to  $\frac{1}{2}BB_1$ . Take a smooth edged slip of paper and mark off on it  $Pa = CA$  and  $Pb = CB$  ( $a$  and  $b$  may be on the same or on opposite sides of  $P$ ). Keep the point  $a$  on the minor axis and the point  $b$  on the major axis and (as already demon-



strated) the point  $P$  will be on the curve. Any number of points may thus be determined. In the lower portion of the figure the

Fig. 55.



lengths  $CA$ ,  $CB$  are shewn set off on opposite sides of  $P$ , and this arrangement is the better when the lengths  $AA_1$ ,  $BB_1$  are nearly equal, as in that case, when set off on the same side of  $P$ , the distance  $ab$  is too short to determine the direction of  $Pa$  with accuracy.

*Second Method* (fig. 56). Arrange the axes as above, and on each as diameter describe a circle. Draw any number of radii as  $C1$ ,  $C2$ , &c. From the extremities of the radii of the circle on the major axis draw lines parallel to the minor axis, and from the ends of the radii of the circle on the minor axis draw lines parallel to the major axis. The lines drawn from corresponding points (as  $7P$ ,  $7'P$ ) will intersect on the required ellipse, which can therefore be drawn through the points thus determined.

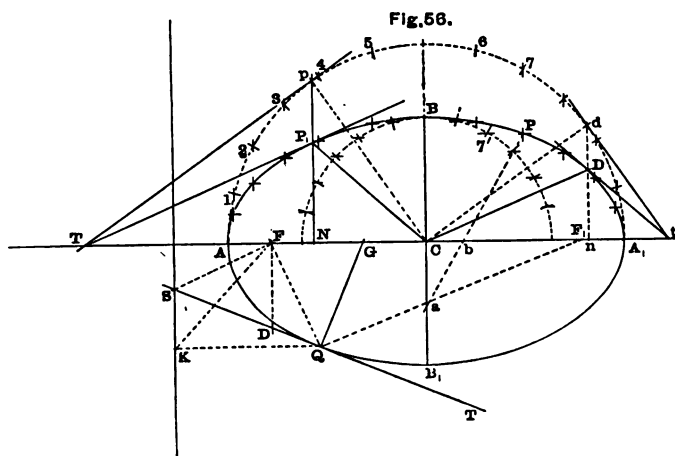
The proof is at once obvious by drawing through any point  $P$  on the curve a line parallel to the corresponding radius  $C7$ , cutting the axes in  $b$  and  $a$ . Then

$CaP7$  is a parallelogram, and  $\therefore Pa = C7 = CA$ ,

$CbP7'$  is a parallelogram, and  $\therefore Pb = C7' = CB$ ,

so that the points found by this construction are identical with those found by the first.

*Third Method* (fig. 56). Determine the foci;—i.e. from the end of the minor axis ( $B_1$ ) as centre describe an arc with radius =  $CA$



cutting  $AA_1$  in  $F$  and  $F_1$ . Stick a pin firmly through the paper at each of the three points  $B_1$ ,  $F$ ,  $F_1$ , and tie a fine thread or piece of silk tightly round these pins, keeping it down in contact with the paper while doing so. Take out the pin at  $B_1$  and keeping the string stretched with the point of a pencil, the curve may be drawn by moving the pencil round the circuit. This method is theoretically perfect, but it fails in practice to give a very exact result chiefly owing to the extensibility of the string and the impossibility of keeping it at a constant tension. It is difficult moreover to tie up the loop of the string to exactly the proper length and to keep the string continually in contact with the paper. Its use therefore cannot be recommended, but it illustrates a very important property of the ellipse, viz. *That the sum of the focal distances of any point on the ellipse is constant and equal to the major axis*, which may be proved thus :



In fig. 55,  $P$  is any point on the ellipse,

$$FP : PK :: FA : AX;$$

also

$$F_1P : PK_1 :: F_1A : AX_1,$$

$$\therefore FP + PF_1 : PK + PK_1 :: FA + F_1A : AX + AX_1,$$

$$: XX_1 :: AA_1 : XX_1,$$

i. e.

$$FP + FP_1 = AA_1.$$

*To draw the tangent at any point of the curve.*

If  $Q_1$  and  $Q_2$  (fig. 55) be any two adjacent points of the curve, and the straight line drawn through them meets a directrix in  $f$ ; draw  $Q_1k_1$ ,  $Q_2k_2$  perpendicular to the directrix and draw  $fF_2$  to the corresponding focus.

Then

$$F_1Q_1 : F_1Q_2 :: Q_1k_1 : Q_2k_2 \\ :: Q_1f : Q_2f,$$

therefore  $F_1f$  bisects the exterior angle between  $Q_1F_1$  and  $Q_2F_1$ . (Euc. VI. Prop. A.)

Hence, exactly as in the case of the parabola (p. 59), when  $Q_2$  moves up to and coincides with  $Q_1$  so that the line through  $Q_1Q_2$  becomes the tangent at  $Q_1$  (Def. p. 30), the line  $F_1f$  becomes perpendicular to the line joining the focus to the point of contact of the tangent. *The tangent at any point  $Q_1$  of an ellipse may therefore be drawn* by drawing a line from  $Q_1$  to either focus, erecting a perpendicular to this line at the focus meeting the directrix, and drawing the tangent through this point and the proposed point of contact. It may also be drawn by using the known property *that the normal bisects the angle between the focal distances*, which may be proved thus. In fig. 56  $Q$  is any point of the curve,  $F$  is a focus, and  $FS$  is perpendicular to  $QF$  meeting the corresponding directrix in  $S$  so that  $QS$  is the tangent at  $Q$ . Draw the normal  $QG$  perpendicular to  $QS$  meeting the major axis in  $G$ , and draw  $FD$  perpendicular to the major axis meeting  $QS$  in  $D$ , and  $QK$  perpendicular to the directrix. Join  $FK$ .

The angle  $QFG$  is the complement of  $QFD$  and is therefore equal to the angle  $SFD$ ; the angle  $FQG$  is the complement of  $SQF$  and is therefore equal to the angle  $FSD$ , and therefore the triangle  $QFG$  is similar to the triangle  $SFD$ .

Hence  $FG : FQ :: FD : FS$  .....(1).

But since  $SFQ, SKQ$  are right angles, a circle can be described round  $FSKQ$ , and therefore the angle  $FSQ =$  the angle  $FKQ$ .

Also the angle  $QFG =$  the angle  $FQK$  since  $GF$  is parallel to  $QK$ , therefore the angle  $FQK = SFD$ , therefore the triangle  $SFD$  is similar to the triangle  $KQF$ , and

$$\therefore FD : FS :: FQ : QK :: FA : AX,$$

$$\therefore FG : FQ :: FA : AX \quad \text{from (1);}$$

similarly  $F_1G : F_1Q :: FA : AX,$

and  $\therefore FG : F_1G :: FQ : F_1Q,$

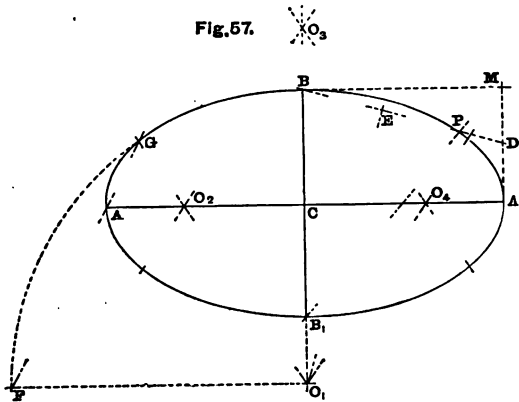
or the angle  $FQF_1$  is bisected by the normal  $QG$ . (Euc. VI. 3.)

Hence  $SQT$  being the tangent the angle  $SQF$  is equal to the angle  $TQF_1$ , or *the tangent is equally inclined to the focal distances of the point of contact*. It follows that if  $F_1Q$  be produced to  $L$  the tangent bisects the angle  $FQL$ .

**PROBLEM 61.** *To describe approximately by means of circular arcs, an ellipse having given axes.*

*First Method* (fig. 57).  $CA, CB, CA_1, CB_1$  are the semi-axes. Draw  $A_1M$  parallel to  $CB$  and  $BM$  parallel to  $CA$  meeting in  $M$ .

Fig. 57.



Bisect  $A_1M$  in  $D$ . Join  $BD$  and draw  $MB_1$  cutting  $BD$  in  $P$ .  $P$  will be a point on the true ellipse with axes  $AA_1$  and  $BB_1$ . Bisect

*PB* in *E*. Draw  $EO_1$  perpendicular to  $PB$  meeting  $BB_1$  in  $O_1$ , and with centre  $O_1$  and radius to  $B$  or  $P$  draw the arc  $PBF$  meeting in  $F$  a line through  $O_1$  parallel to  $AA_1$ . Draw  $FA$  and produce it to meet the arc in  $G$ . Draw  $GO_1$  cutting  $AA_1$  in  $O_2$ , and with centre  $O_2$  and radius  $O_2G$  draw an arc which will be found to pass through  $A$ , since by the similar triangles  $GO_2A$ ,  $GO_1F$ ,

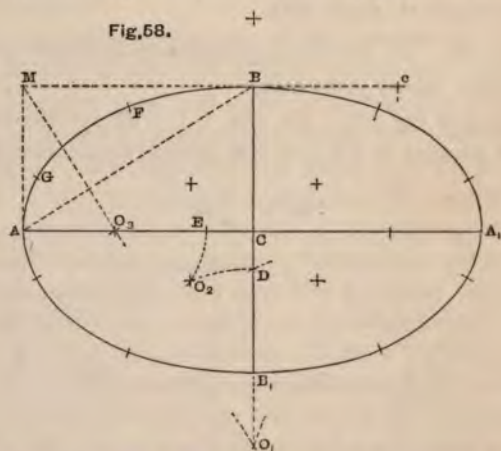
$$GO_2 : AO_2 :: GO_1 : FO_1,$$

$$\text{i.e. } O_2G = O_2A.$$

The two arcs  $AG$  and  $GB$  form one quadrant of the approximate ellipse and the remainder can of course be put in by symmetry, taking centres  $O_3$  and  $O_4$  in corresponding positions to those already obtained.

*Second Method* (fig. 58). Draw  $AM$ ,  $BM$  parallel respectively to  $BC$ ,  $AC$ , meeting in  $M$ . Draw  $MO_1$  perpendicular to  $AB$ ,

Fig. 58.



cutting  $BB_1$  in  $O_1$  and  $AA_1$  in  $O_3$ . Find a mean proportional ( $BD$ ) between  $CA$  and  $CB$ . (This may conveniently be done by making  $Bc$  on  $MB$  produced equal to  $BC$ , and describing a semi-circle on  $Mc$  cutting  $BC$  in  $D$ .) Make  $AE$  equal to  $BD$ . With centres  $O_1$ ,  $O_3$  and radii  $O_1D$ ,  $O_3E$  describe arcs intersecting in  $O_2$ . Then  $O_1$ ,  $O_2$ ,  $O_3$  are points which can be used as centres for

successive arcs of the required curve. The arc struck from  $O_1$  will pass through  $B$  and extend of course to  $F$  on the line  $O_1O_2$ , that from  $O_2$  will pass through  $F$  and extend to  $G$  on  $O_2O_3$ , and that from  $O_3$  will start from  $G$  and pass through  $A$ . Thus each quadrant will consist of three arcs, and the centres for the other three quadrants can be taken by symmetry.

The arc struck with centre  $O_2$  and radius  $O_2G$  will evidently pass through  $A$ , since  $GO_2 = FO_2 = BD = AE$  and

$$GO_2 = GO_2 - O_2O_3 = GO_2 - O_2E = AE - O_2E = AO_2.$$

It will be shewn hereafter that the points  $O_1, O_2$  are the *centres of curvature* at  $B$  and  $A$  respectively; the circular arcs struck with these centres through  $B$  and  $A$  coincide therefore more nearly with the true ellipse at those points than any others which can be drawn.

**DEFINITION.** Any line drawn through the centre of the ellipse and terminated both ways by the curve is called a *diameter*, and a semi-diameter  $CD$  parallel to the tangent at the extremity of a semi-diameter  $CP$  is said to be *conjugate to  $CP$* . Every diameter is evidently bisected by the centre.

The following important properties of the ellipse should be carefully noticed.

**PROP. 1.** *Tangents drawn at the extremities of any chord subtend equal angles at the focus.*

Let  $PP_1$  (fig. 59) be any chord of an ellipse, and let the tangents at  $P$  and  $P_1$  meet in  $T$ . Let  $F$  be the focus, and from  $T$  draw  $TM, TM_1$  perpendicular to  $FP, FP_1$ , and draw  $TN$  perpendicular to the directrix  $XS$ . Let the tangent at  $P_1$  meet the directrix in  $S$ , then  $FS$  is perpendicular to  $FP_1$  and therefore parallel to  $TM_1$ ,

$$\begin{aligned} \therefore FM_1 : FP_1 &:: ST : SP_1 \\ &:: TN : P_1K, \end{aligned}$$

where  $P_1K$  is perpendicular to the directrix;

$$\begin{aligned} \therefore FM_1 : TN &:: FP_1 : P_1K \\ &:: FA : AX. \end{aligned}$$

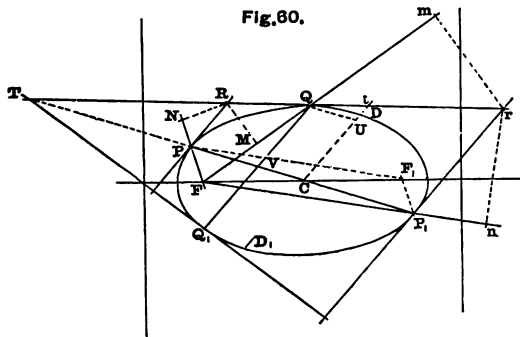


Take any second chord  $qq_1$  parallel to  $QQ_1$ , meeting  $F_1Y$  in  $Y_1$  and the directrix in  $R_1$ , let  $O_1$  be its centre point, then since  $\frac{OY}{OR} = \frac{O_1Y_1}{O_1R_1}$ , it follows that the line  $OO_1$  must pass through the point  $T_1$  in which  $F_1Y$  meets the directrix and is therefore fixed for all chords parallel to  $QQ_1$ . This line  $T_1O$  will pass through the centre (i.e. will be a diameter), because the chord through the centre parallel to  $QQ_1$  is bisected by the centre and also by  $T_1O$ . Let  $T_1O$  meet the ellipse in  $P_s$  and suppose  $qq_1$  to move parallel to itself till it approaches and ultimately passes through  $P_s$ . Since  $O_1q = O_1q_1$  throughout the motion the points  $q, q_1$  will evidently approach  $P_s$  simultaneously, and in the limiting position  $qq_1$  will be the tangent at  $P_s$ . It follows that if  $P_s$  be the other extremity of the diameter through  $P_s$ , the tangent at  $P_s$  is parallel to  $QQ_1$ , and therefore to the tangent at  $P_s$ .

**COROLLARY.** *The perpendicular on the tangent at any point from the focus meets the corresponding diameter in the directrix.*

**PROP. 3.** *If  $PCP_1$  be a diameter and  $QVQ_1$  a chord parallel to the tangent at  $P$  and meeting  $PP_1$  in  $V$ , and if the tangent at  $Q$  meet  $PP_1$  produced in  $T$ , then  $CV \cdot CT = CP^2$  (fig. 60).*

Let  $TQ$  meet the tangents at  $P$  and  $P_1$  in  $R$  and  $r$ , and  $F$  being a focus draw  $RN$  perpendicular to the focal distance  $FP$



meeting  $FP$  in  $N$ ,  $rn$  perpendicular to  $FP_1$  meeting it in  $n$ , and  $RM, rn$  perpendicular to the focal distance  $FQ$ .



Let  $F_1$  be the other focus and join  $F_1P$ ,  $F_1P_1$ .

Since  $CF = CF_1$ ,  $CP = CP_1$ , and the angle  $F_1CP =$  the angle  $F_1CP_1$ , therefore the triangles  $F_1CP$ ,  $F_1CP_1$  are equal in all respects, and therefore the angle  $CPF =$  the angle  $CP_1F_1$ ;

similarly

$$CPF_1 = \text{the angle } CP_1F,$$

and therefore the whole angle  $FPF_1 =$  the whole angle  $F_1P_1F$ .

But the tangents are equally inclined to the focal distances, and therefore also the angle  $FPR =$  the angle  $F_1P_1r$ ,

$$\therefore \text{the angle } FPR = \text{the angle } FP_1r,$$

i.e. the right-angled triangles  $RPN$ ,  $rP_1n$  are similar, and therefore

$$RP : rP_1 :: RN : rn.$$

But

$$RN = RM \text{ and } rn = rm \text{ (Prop. 1),}$$

$$\therefore RP : rP_1 :: RM : rm$$

$$:: RQ : rQ.$$

But

$$TR : Tr :: RP : rP_1,$$

$$\therefore TP : TP_1 :: PV : P_1V,$$

or

$$CT - CP : CT + CP :: CP - CV : CP + PV,$$

$$\therefore CT : CP :: CP : CV,$$

$$\therefore CT \cdot CV = CP^2.$$

*Cor. 1.* Since  $CV$  and  $CP$  are the same for the point  $Q_1$ , the tangent at  $Q_1$  passes through  $T$  or the tangents at the extremities of any chord intersect on the diameter which bisects that chord.

*Cor. 2.* Since  $TP_1 : TP :: P_1V : VP$ , it follows that  $TPVp$  is harmonically divided (p. 13).

The above proposition has been proved generally; it therefore holds when the diameter  $CP$  coincides with the major axis. Let  $P_1$  be any point on an ellipse (fig. 56) and draw the ordinate  $P_1N$  perpendicular to  $CA$ , producing it to meet the auxiliary circle in  $p$ , and draw the tangent at  $P_1$  meeting  $CA$  in  $T$ , then

$$CN \cdot CT = CA^2 = Cp^2,$$

and

$$\therefore CpT \text{ is a right angle,}$$

and therefore  $pT$  is a tangent at  $p$  to the auxiliary circle: hence

*Cor. 3. The tangents at the extremities of corresponding ordinates of the ellipse and auxiliary circle intersect on the major axis.*

Draw  $CD$  (fig. 56) the diameter conjugate to  $CP_1$ ,  $dDn$  the corresponding ordinate meeting the auxiliary circle in  $d$ , and the tangents at  $D$  and  $d$  meeting the major axis in  $t$ .

Then  $P_1N : pN :: BC : AC :: Dn : dn$ ,  
and  $P_1N : NT :: Dn : Cn$ ,  
since  $CD$  is parallel to  $P_1T$ ,

$$\therefore pN : NT :: dn : Cn,$$

and therefore  $Cd$  is parallel to  $pT$ , i.e.  $pCd$  is a right angle, or

*Cor. 4. Conjugate diameters in the ellipse project into diameters at right angles to each other in the auxiliary circle.*

If the tangent at  $d$  meet the major axis in  $t$ , since  $dt$  is parallel to  $Cp$ ,  $Dt$  (the tangent at  $D$ ) will be parallel to  $CP_1$ , or,

*Cor. 5. If  $CD$  be conjugate to  $CP_1$ ,  $CP_1$  is also conjugate to  $CD$ .*

Since  $pCd$  is a right angle, the angle  $dCn$  is the complement of the angle  $pCN$ , and therefore equals the angle  $CpN$ , therefore the triangles  $CpN$ ,  $dCn$  are equal in all respects, i.e.  $Cn = pN$  and  $dn = CN$ ,

$$\begin{aligned} CP_1^2 &= P_1N^2 + CN^2 \text{ and } CD^2 = Cn^2 + Dn^2, \\ \therefore \overline{CP_1}^2 + \overline{CD}^2 &= \overline{CN}^2 + \overline{pN}^2 + \overline{P_1N}^2 + \overline{Dn}^2 \\ &= \overline{CA}^2 + \overline{P_1N}^2 + Dn^2 \dots\dots\dots(1). \end{aligned}$$

But  $P_1N : pN :: Dn : dn :: BC : AC$ ,

$$\therefore P_1N : Dn :: pN : dn,$$

$$\therefore P_1N^2 + Dn^2 : P_1N^2 :: pN^2 + dn^2 : pN^2,$$

and  $pN^2 + dn^2 = AC^2$ ;

or  $P_1N^2 + Dn^2 : \overline{AC}^2 :: BC^2 : AC^2$ ,

$$\therefore \overline{P_1N}^2 + Dn^2 = BC^2.$$

Therefore, from (1)

*Cor. 6.  $\overline{CP_1}^2 + \overline{CD}^2 = \overline{CA}^2 + \overline{CB}^2$ .*



PROP. 4. If  $PCP_1$ ,  $DCD_1$  be conjugate diameters and  $QV$  be drawn parallel to  $CD$  meeting the ellipse in  $Q$  and  $CP$  in  $V$ , then

$$QV^2 : PV \cdot VP_1 :: CD^2 : CP^2.$$

[ $QV$  is called an ordinate of the diameter  $PCP_1$ .]

Let the tangent at  $Q$  (fig. 60) meet  $CP$ ,  $CD$  in  $T$  and  $t$ , and draw  $QU$  parallel to  $CT$  meeting  $CD$  in  $U$ .

Then  $CV \cdot CT = CP^2$  and  $CU \cdot Ct = CD^2$  (Prop. 3).

But

$$CU = QV,$$

$$\therefore CD^2 : CP^2 :: QV \cdot Ct : CV \cdot CT$$

$$:: \overline{QV}^2 : CV \cdot VT.$$

Since

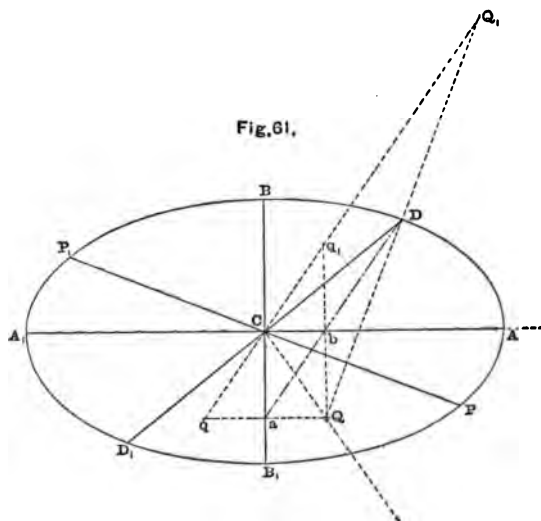
$$Ct : QV :: CT : VT,$$

and

$$\begin{aligned} CV \cdot VT &= CV \cdot CT - CV^2 \\ &= CP^2 - CV^2 = PV \cdot VP_1, \end{aligned}$$

$$\therefore QV^2 : PV \cdot VP_1 :: CD^2 : CP^2.$$

PROBLEM 62. *Given a pair of conjugate diameters to determine the axes* (fig. 61).



$PCP_1$ ,  $DCD_1$  are the given conjugate diameters. Through  $D$  draw  $Q_1DQ$  perpendicular to  $CP$ . Make  $DQ$  and  $DQ_1$  each equal to  $CP$  and draw the lines  $CQ$ ,  $CQ_1$ . Then the major axis  $ACA_1$  bisects the angle  $QCQ_1$ , and the minor axis ( $BCB_1$ ) is of course a line through  $C$  perpendicular to  $ACA_1$ . The axes are therefore determined in direction. To determine them in magnitude:—On  $Q_1C$  on opposite sides of  $C$  make  $Cq$  and  $Cq_1$  each equal to  $CQ$ , then  $Q_1q$  will be the length of the major axis  $AA_1$  and  $Q_1q_1$  will be the length of the minor  $BB_1$ . Bisect each of these lines and  $CA$ ,  $CB$  will be given respectively.

*Proof.* Since  $CQ = Cq$  and  $BC$  bisects the angle  $QCq$ , therefore  $a$ , the point in which  $Qq$  cuts  $BC$ , bisects  $Qq$  and therefore  $Da$  is parallel to  $Q_1q$  and  $= \frac{Q_1q}{2} = CA$ . Similarly  $b$  the point in which  $Qq_1$  meets  $CA$  bisects  $Qq_1$  and  $Db$  is parallel to  $Q_1q_1$  and  $= \frac{Q_1q_1}{2} = CB$ , and  $D$ ,  $b$ ,  $a$  are in the same straight line. Hence  $D$  is a point on the ellipse described with  $CA$ ,  $CB$  as semi-axes. Also  $DQ$  is the normal at  $D$ , since  $Q$  is the instantaneous centre of rotation for the line  $ab$  moving along the axes. Therefore the tangent at  $D$  will be parallel to  $CP$ . Lastly, to shew that  $P$  will also be on the curve,

$$\begin{aligned} CQ^2 + CQ_1^2 &= 2CD^2 + 2DQ^2 \text{ (Euc. II. 12 and 13),} \\ &= 2(CD^2 + CP^2). \end{aligned}$$

But

$$CQ_1 = AC + BC,$$

and

$$CQ = AC - BC,$$

$$\therefore CQ^2 + CQ_1^2 = 2(AC^2 + BC^2),$$

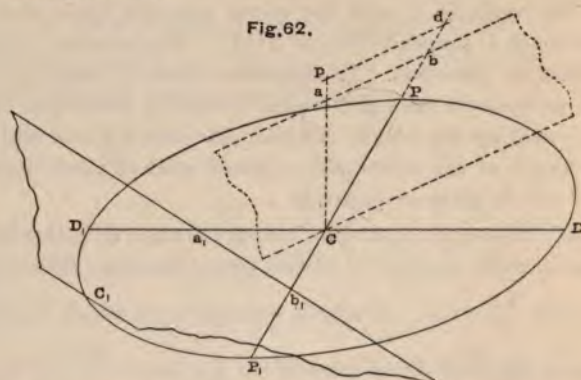
$$\therefore CD^2 + CP^2 = AC^2 + BC^2,$$

a known property of conjugate diameters. (See Cor. 6, p. 109.)

**PROBLEM 63.** *To describe an ellipse having given conjugate diameters  $PCP_1$ ,  $DCD_1$ .*

This might of course be done by the last problem: the curve may however be drawn independently, though none of the following constructions give any information as to the position of the axes, foci, or directrices.

*First Method.* By continuous motion (fig. 62). From  $C$  draw  $Ca$  perpendicular to  $CD$  and through  $P$  draw  $Pa$  parallel to  $CD$  meeting  $Ca$  in  $a$ .



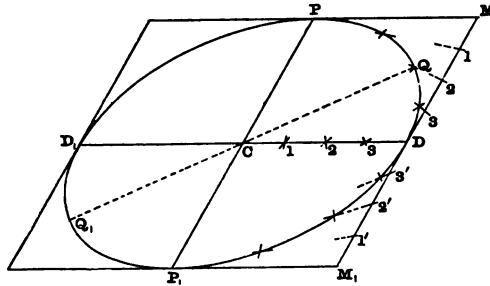
On  $CP$  make  $Cd = CD$  and on  $Ca$  make  $Cp = CP$ . Through  $a$  draw  $ab$  parallel to  $pd$  meeting  $CP$  in  $b$ . If a triangle equal and similar to the triangle  $abC$  be moved round so that the angle  $a$  is always on the diameter  $DCD_1$ , and the angle  $b$  on  $PCP_1$ , the angle  $C$  will be on the curve. The most convenient way of proceeding practically is to cut a strip of paper of breadth equal to the perpendicular distance between  $C$  and  $ab$ . The points  $a$  and  $b$  can then be marked off on one edge (as at  $a_1b_1$ ) and the point  $C$  on the other edge (as at  $C_1$ ). The slip can easily be adjusted in any number of positions and the corresponding positions of  $C_1$  marked. Any number of points on the curve may thus be determined\*.

*Second Method* (fig. 63). Draw  $PM$ ,  $P_1M_1$  parallel to  $CD$  and  $DMM_1$  parallel to  $CP$  meeting  $PM$  in  $M$  and  $P_1M_1$  in  $M_1$ . Divide  $MD$  into any number of equal parts 1, 2, 3... and  $CD$  into the same number of equal parts. Then lines drawn from  $P$  to any of the points on  $MD$  intersect lines drawn from  $P_1$  through the corresponding points on  $CD$  in points on the curve, and thus any number of points in the quadrant  $PD$  can be determined.

\* I am indebted to Prof. Minchin for this construction.

Similarly if  $M_1D$  be divided into any number of equal parts  $1', 2', 3'...$  and  $CD$  into the same number  $1, 2, 3...$  lines drawn from  $P_1$  to the points on  $M_1D$  intersect lines drawn from  $P$  to

Fig. 63.



the corresponding points on  $CD$  in points on the required curve, and thus the quadrant  $DP_1$  can be determined. When half the curve is drawn the remainder can be put in by symmetry, since every diameter is bisected by the centre; thus if  $QCQ_1$  be drawn and  $CQ_1$  be made equal to  $CQ$ ,  $Q_1$  will be a point on the curve and similarly for any other points on the semi-ellipse  $P, D_1, P_1$ .

*Third Method* (fig. 64).  $PCP_1, DCD_1$  are the given conjugate diameters. Draw  $PM, P_1M_1$  parallel to  $CD$  and  $MDM_1$  parallel to  $CP$  meeting  $PM$  in  $M$  and  $P_1M_1$  in  $M_1$ .

Draw the line  $PD$  and take on it any number of points  $1, 2, 3...$

Draw the lines  $1a, 2b, 3c...$  parallel to  $CD$  meeting  $DM$  in  $a, b, c...$ ; and the lines  $M_11, M_12, M_13...$  meeting  $MP$  in  $a', b', c'...$ . Then the lines  $aa', bb', cc'...$  will be tangents to the curve, which must be drawn in to touch these lines, so giving the quadrant  $PD$ . A similar construction will give a second quadrant  $DP_1$ , and the remaining semi-ellipse can of course be put in similarly or by drawing any number of diameters.

*Fourth Method* (fig. 64). Draw  $PM_2, P_1M_2E$  parallel to  $CD$ , and  $M_2D_1M_2$  parallel to  $PP_1$  meeting  $PM_2$  in  $M_2$  and  $PM_2$  in  $M_2$ . Make  $M_2E = P_1M_2$  and divide  $PM_2$  into any number of equal parts as at  $1, 2, 3...$ . Draw  $E1, E2, E3...$  cutting  $D_1M_2$  in  $f, g, h$

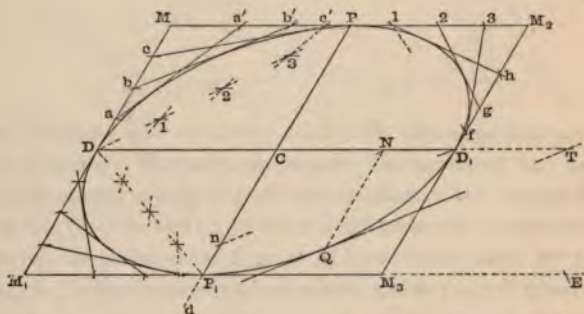
respectively. Then the lines joining corresponding points on  $PM_2$  and  $M_2D_1$ , as  $f3, g2$  and so on, will be tangents to the curve, which must therefore be drawn touching these lines.

Similarly for the remaining quadrants, or as before, when half the ellipse is obtained the other can be put in by symmetry.

*To draw a tangent at any point of an ellipse having a given pair of conjugate diameters.*

Let  $Q$  (fig. 64) be the point,  $PCP_1, DCD_1$  the given conjugate diameters. Draw  $QN$  parallel to  $CP$  meeting  $CD_1$  in  $N$ , so that

Fig. 64.



$CN$  is the abscissa and  $QN$  the ordinate of  $Q$  referred to the given conjugate diameters as axes. Make  $Cn$  on  $CP_1$  equal to  $CN$  and  $Cd = CD_1$  and draw through  $d$  a line  $dT$  parallel to  $nD_1$  cutting  $CD_1$  in  $T$ . The line  $QT$  will be the tangent at  $Q$ , for by similar triangles

$$CT : Cd :: CD_1 : Cn,$$

$$\text{i.e. } CT : CD_1 :: CD_1 : CN \text{ (Prop. 3, p. 107).}$$

**PROBLEM 64.** *To describe an ellipse, one axis and a point ( $P$ ) on the curve being given (Fig. 55).*

The axis is of course given in direction and magnitude, and this really involves the centre of the curve and the position of the other axis.

First, suppose the major axis  $AA_1$  given. Bisect it in  $C$  and draw  $BCB_1$  perpendicular to  $AA_1$ . From  $P$  with  $AC$  as radius

mark the point  $a$  on  $BB_1$  and draw  $Pa$  cutting  $AA_1$  in  $b$ .  $Pb$  will be the length of the semi-minor axis, which can therefore be marked off from  $C$  to  $B$  and  $B_1$ .

Second, if the minor axis  $BB_1$  is given. Bisect it in  $C$ , through  $C$  draw  $ACA_1$  perpendicular to  $BB_1$ . From  $P$  with radius  $BC$  mark off the point  $b$  on  $AA_1$ , and draw  $Pb$ , producing it to meet  $BB_1$  in  $a$ . Then  $Pa$  will be the length of the semi-major axis, which can be set off from  $C$  to  $A$  and  $A_1$ .

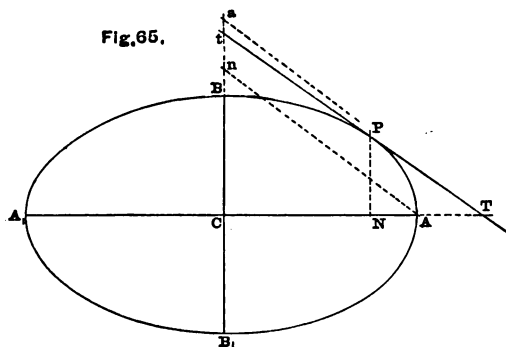
The construction is obvious from the original method of drawing the curve.

**PROBLEM 65.** *To describe an ellipse, an axis  $ACA$ , and a tangent  $Tt$  being given (Fig. 65).*

$T, t$  are the points in which the given tangent cuts the axes.

**Draw the second axis  $BCB_1$ .**

Take  $CN$  on  $CA$ , a third proportional to  $CT, CA$  (i.e. on  $CB$  make  $Ca=CA$ , draw  $An$  parallel to  $Ta$ , cutting  $CB$  in  $n$ , and



make  $CN = Cn$ ). Then  $N$  is the foot of the ordinate of the point of contact of the given tangent (Prop. 3, p. 107), and therefore by drawing  $NP$  perpendicular to  $CA$  meeting the tangent in  $P$ , a point on the curve is determined and the problem reduces to Problem 64.

PROBLEM 66. *To describe an ellipse, the directions of a pair of conjugate diameters  $CA$ ,  $CB$ , a tangent  $PT$  and its point of contact  $P$  being given (Fig. 65).*

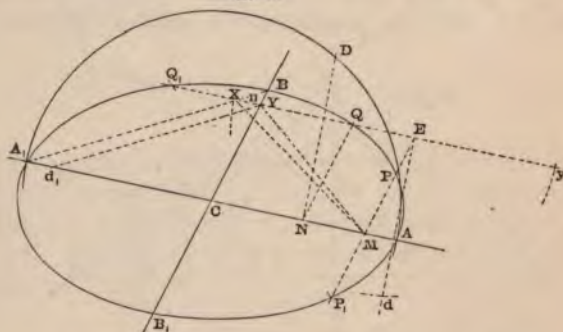
In the figure the given conjugate diameters are the axes, but the construction holds in any case.

Through  $P$  draw  $PN$  parallel to  $CB$  meeting  $CA$  in  $N$ . Take  $CA$  a mean proportional between  $CN$  and  $CT$ , which determines the length of the semi-diameter  $CA$ . Similarly determine the length  $CB$ .

PROBLEM 67. *To describe an ellipse, the centre ( $C$ ), two points on the curve ( $P$  and  $Q$ ), and the directions of a pair of conjugate diameters ( $CA$ ,  $CB$ ) being given. The lengths  $CA$ ,  $CB$  are not given (Fig. 66).*

From  $P$  and  $Q$  draw  $PM$ ,  $QN$  parallel to  $BC$  meeting  $CA$  in  $M$  and  $N$ . [In order that the problem may be possible, if  $PM$  is

Fig. 66.



less than  $QN$ ,  $CM$  must be greater than  $CN$ .] Produce  $PM$  to  $E$  and  $P_1$ , making  $MP_1$  equal to  $MP$ ;  $P_1$  will evidently be a point on the curve. Similarly, drawing  $EQnQ_1$  parallel to  $CA$  meeting  $PM$  in  $E$  and  $CB$  in  $n$ , and making  $nQ_1 = nQ$ ,  $Q_1$  will be a point on the curve. Through  $M$  draw  $MX$  parallel to  $PQ$  and  $MY$  parallel to  $P_1Q_1$  meeting  $EQ_1$  in  $X$  and  $Y$  respectively. Find  $ND$  a mean proportional between  $EX$  and  $EY$  and set it up from  $N$  on a perpendicular to  $CA$ . [The mean proportional may con-

veniently be found by producing  $XE$  to  $y$ , making  $Ey = EY$ , and on  $Xy$  describing a semi-circle cutting  $Ed$  perpendicular from  $E$  to  $Xy$  in  $d$ .  $Ed$  is the required mean proportional.] Then a circle described with centre  $C$  and radius  $CD$  cutting  $CA$  in  $A$  and  $A_1$ , will determine  $A$  and  $A_1$  the extremities of that diameter, and if  $Cd_1$  be made  $= ND$  on  $CA_1$ , and a parallel to  $d_1n$  be drawn through  $A_1$  cutting  $CB$  in  $B$ , this will determine an extremity of the other. The curve can then be completed by preceding problems.

*Proof.* The construction depends on the known proposition that  $EP \cdot EP_1 : EQ \cdot EQ_1 :: CB^2 : CA^2$ ;  $PP_1$  and  $QQ_1$  being any chords parallel to the conjugate diameters  $CB$ ,  $CA$  and intersecting in  $E$ . Admitting this, then by Prop. 4, p. 110,

$$QN^2 : AN \cdot NA_1 :: EP \cdot EP_1 : EQ \cdot EQ_1.$$

By the construction

$$EX : EQ :: QN : EP,$$

and

$$EY : EQ_1 :: QN : EP_1,$$

$$\therefore EX \cdot EY : EQ \cdot EQ_1 :: QN^2 : EP \cdot EP_1;$$

but

$$EX \cdot EY = ND^2 = AN \cdot NA_1,$$

$$\therefore AN \cdot NA_1 : QN^2 :: EQ \cdot EQ_1 : EP \cdot EP_1,$$

which proves that  $AA_1$  is the diameter parallel to  $EQQ_1$ .

Also by construction

$$CB : CA_1 :: QN : ND;$$

$$\therefore CB^2 : CA_1^2 :: QN^2 : AN \cdot NA_1,$$

or  $CB$  is the semi-diameter conjugate to  $CA_1$ .

That  $EP \cdot EP_1 : EQ \cdot EQ_1 :: CB^2 : CA^2$  may be proved thus:—

Through  $E$  draw the diameter  $ERR_1$  and draw the ordinate  $RU$  parallel to  $PP_1$  or to  $CB$ , then by Prop. 4, p. 110,

$$CB^2 - RU^2 : CU^2 :: CB^2 : CA^2,$$

and

$$PM^2 : CA^2 - CM^2 :: CB^2 : CA^2;$$

$$\therefore CB^2 - PM^2 : CM^2 :: CB^2 : CA^2,$$

so that

$$CB^2 - RU^2 : CU^2 :: CB^2 - PM^2 : CM^2;$$

but

$$RU^2 : CU^2 :: EM^2 : CM^2;$$

$$\therefore CB^2 : CU^2 :: CB^2 - PM^2 + EM^2 : CM^2,$$

$$\text{or } CB^2 : CB^2 - PM^2 + EM^2 :: CU^2 : CM^2 :: CR^2 : CE^2;$$

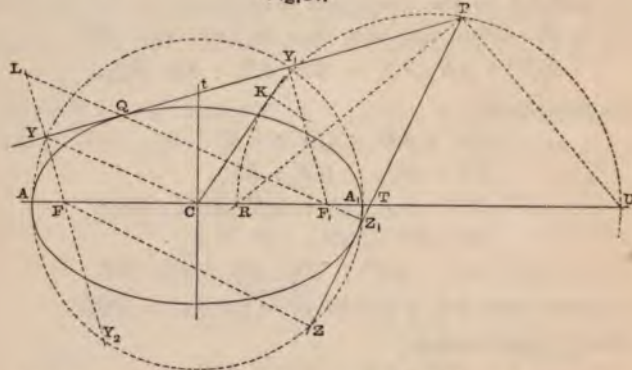


$$\begin{aligned} \therefore CB^2 : EM^2 - PM^2 &:: CR^2 : CE^2 - CR^2, \\ \text{or} \quad CB^2 : EP \cdot EP_1 &:: CR^2 : ER \cdot ER_1, \\ \text{Similarly} \quad CA^2 : EQ \cdot EQ_1 &:: CR^2 : ER \cdot ER_1, \\ \text{or} \quad EP \cdot EP_1 : EQ \cdot EQ_1 &:: CB^2 : CA^2. \end{aligned}$$

PROBLEM 68. To describe an ellipse, the centre ( $C$ ), direction of major axis  $CT$ , and two tangents ( $PT$ ,  $Pt$ ) being given (Fig. 67).

Bisect the angle  $TPt$  between the given tangents by  $PR$  meeting  $CT$  in  $R$ , and draw  $PU$  perpendicular to  $PR$  meeting  $CT$  in  $U$ .

Fig. 67.



Describe a circle round the triangle  $RPU$  and draw a tangent from  $C$  to this circle meeting it in  $K$ .  $CK$  will be the distance of either focus from  $C$ , i.e. make  $CF = CF_1 = CK$ , and  $F$  and  $F_1$  will be the foci of the required ellipse. From  $F$  draw  $FY$  perpendicular to  $Pt$  meeting it in  $Y$ , and make  $YL$  on  $FY$  produced  $= YF$ . Draw  $F_1L$  cutting  $Pt$  in  $Q$ , and  $Q$  will be the point of contact of  $Pt$ , i.e.  $Q$  will be a point on the ellipse, which can therefore be completed by preceding problems.

*Proof.* Since  $CK$  is a tangent to the circle  $RPU$ ,

$$CK : CR :: CU : CK;$$

$$\therefore CK + CR : CK - CR :: CU + CK : CU - CK,$$

or

$$FR : RF_1 :: FU : F_1U,$$

i.e.  $FU$  is divided harmonically in  $R$  and  $F_1$  or  $P\{FRF_1U\}$  is

a harmonic pencil. But the angle  $RPV$  is a right angle, and therefore  $PF$  and  $PF_1$  make equal angles with  $PR$  (p. 15). Therefore also the angle  $FPQ =$  the angle  $F_1PT$  since  $PR$  bisects the angle  $QPT$ ; or *the tangents from  $P$  make equal angles with the focal distances of  $P$* : a known property of the ellipse.

$F_1L$  is evidently the length of the major axis, for, by the construction  $QL = FQ$ , and therefore  $F_1L = F_1Q + QF$ , the sum of the focal distances (Prob. 60, p. 101).

It follows that  $Y$  is on the auxiliary circle, for  $CF = CF_1$  and  $FY = YL$ ; therefore  $CY$  is parallel to and equal to  $\frac{1}{2}F_1L = CA$ : and similarly if  $F_1Y_1$ ,  $FZ$  and  $F_1Z_1$  are perpendiculars from the foci on the tangents,  $Y_1$ ,  $Z$  and  $Z_1$  are all on the auxiliary circle. Produce  $YF$  to meet the auxiliary circle in  $Y_2$ , then  $FY_2$  is equal to  $F_1Y_1$ , and therefore

$$FY \cdot F_1Y_1 = FY \cdot FY_2 = AF \cdot FA_1. \quad (\text{Euc. III. 35.})$$

$$\text{Similarly} \quad FZ \cdot F_1Z_1 = AF \cdot FA_1 = FY \cdot F_1Y_1,$$

i. e.

$$FY : FZ :: F_1Z_1 : F_1Y_1;$$

and since the angle  $YFZ$  is equal to the angle  $Y_1F_1Z_1$ , therefore the triangles  $YFZ$ ,  $Z_1F_1Y_1$  are similar (Euc. vi. 6), i. e. the angle

$$FZY = F_1Y_1Z_1.$$

Circles can be described about the figures

$$YFZP \text{ and } F_1Z_1PY_1,$$

and therefore the angle  $FPY =$  the angle  $FZY$ ,

$$,, \quad ,, \quad F_1PZ_1 = ,, \quad F_1Y_1Z_1; \quad (\text{Euc. III. 21.})$$

therefore the angle  $FPY =$  the angle  $F_1PZ_1$ , which proves the property above referred to.

**PROBLEM 69.** *To describe an ellipse, the centre  $C$ , the directions of a pair of conjugate diameters  $CT$ ,  $Ct$ , a tangent  $Tt$ , and a point  $P$  being given (Fig. 68).*

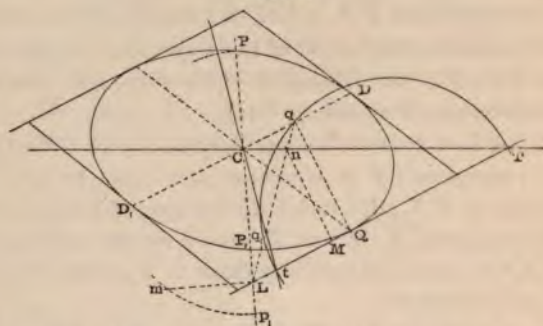
[ $P$  must lie between the line  $Tt$  and a parallel corresponding line on the other side of  $C$ .]

Draw  $PCL$  meeting  $Tt$  in  $L$ , and make  $CP_1 = CP$ .  $P_1$  is a point on the curve,

Take a mean proportional ( $Lm$ ) between  $LP$  and  $LP_1$  and

make  $LM$  on  $LT$  equal to  $Lm$ . On  $Tt$  describe a semicircle  $Tqq,t$ ; draw  $Mn$  perpendicular to  $LM$  and make  $Mn = CP$ . Draw  $Ln$  cutting the semicircle in  $q$  and  $q_1$ . From  $q$  draw  $qQ$

Fig. 63.



perpendicular to  $LT$  meeting it in  $Q$ ; then  $Q$  will be the point of contact of  $Tt$ , and  $Qq$  will be the length of  $CD$  the semi-diameter conjugate to  $CQ$ ; the curve can therefore be completed by Problems 62 or 63.

Since  $Ln$  cuts the semicircle in two points, there are two solutions.

*Proof.* The construction depends on the property of the ellipse proved in Problem 67, that the rectangles contained by the segments of intersecting chords are in the ratio of the squares of the parallel diameters; and on the further property that if the tangent at  $Q$  meet a pair of conjugate diameters in  $T$  and  $t$ , and  $CD$  be conjugate to  $CQ$ ,

$$QT \cdot Qt = CD^2.$$

If  $Q$  be the point of contact of  $Tt$  it follows that

$$LP \cdot LP_1 : LQ^2 :: CP^2 : CD^2;$$

but  $LP \cdot LP_1 = LM^2$  by construction,

$$\therefore LM : LQ :: CP : CD;$$

but by construction

$$LM : LQ :: Mn : Qq,$$

and  $Mn = CP$ , so that  $Qq = CD$ ; also  $Qq^2 = QT \cdot Qt$ , since  $Tqt$  is a semicircle, therefore  $CD^2 = QT \cdot Qt$  in the figure as drawn.



Draw  $RCR_1$  and produce it to meet  $TQ$  in  $L$ ; make  $CR_1 = CR$ . Take a mean proportional ( $Lm$ ) between  $LR$  and  $LR_1$  and make  $LM$  on  $TQ = Lm$ . Draw  $Mr_1$  perpendicular to  $TQ$  and equal to  $CR$ , and join  $Lr_1$ , cutting the circle described on  $Tt$  as diameter in  $q$  and  $q_1$ ; from  $q$  or  $q_1$  drop a perpendicular ( $qQ$ ) on  $Tt$ , and  $Q$  will be the point of contact of  $Tt$ .  $CD$  drawn parallel to  $Tt$  and equal to  $Qq$  will be the semi-diameter conjugate to  $CQ$ .

*Proof.* By construction,

$$LM^2 : LQ^2 :: Mr_1^2 : Qq^2,$$

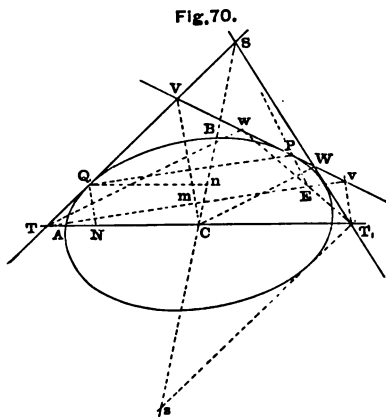
$$\text{i. e. } LR \cdot LR_1 : LQ^2 :: CR^2 : Qq^2;$$

therefore if  $Q$  is the point of contact of  $Tt$ ,  $Qq$  must be the length of the semi-diameter parallel to  $Lt$ : and since

$$QT \cdot Qt = Qq^2 = CD^2,$$

$Q$  is such point of contact. (See last problem.)

**PROBLEM 71.** To describe an ellipse, the centre  $C$  and three tangents ( $SV$ ,  $SW$ ,  $VW$ ) being given (Fig. 70).



Through  $C$  draw  $TCT_1$ , meeting  $SV$  in  $T$  and  $SW$  in  $T_1$ , so that  $TT_1$  is bisected in  $C$  (Prob. 14, p. 19).  $CT$  will be conjugate to  $CS$ . Draw  $T_1v$  parallel to  $CV$  meeting  $VW$  in  $v$ , then  $Tv$  will be an ordinate of the diameter  $CV$ , for if it meets  $CV$  in  $m$ ,  $Tm = mv$ , since  $TC = CT_1$ .

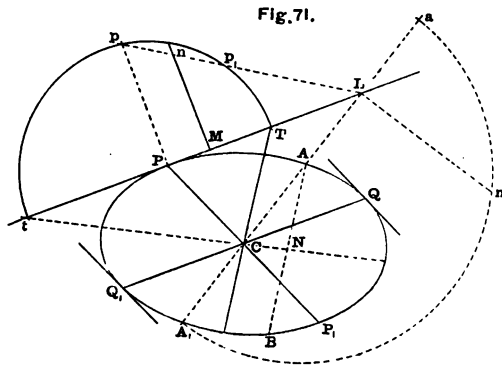
Similarly, if  $Tw$  be drawn parallel to  $CW$  meeting  $VW$  in  $w$ ,  $T_1w$  will be an ordinate of the diameter  $CW$ .

Let  $Tv$ ,  $T_1w$  intersect in  $E$ . Draw  $SE$  cutting  $VW$  in  $P$ , and  $P$  will be the point of contact of  $VW$ . Also  $PQ$  parallel to  $Tv$  meeting  $ST$  in  $Q$  will be the chord of contact of the pair of tangents  $VT$ ,  $VW$ , i.e.  $P$  and  $Q$  are points on the curve; and the problem reduces to one of several previously given, or may be completed thus:—Draw  $QN$  parallel to  $CS$  meeting  $CT$  in  $N$ .  $QN$  is an ordinate of the diameter  $CT$ , and therefore  $CA$  the length of the semi-diameter is a mean proportional between  $CN$  and  $CT$  (Prop. 3, p. 107). Similarly if  $Qn$  be drawn parallel to  $CT$  meeting  $CS$  in  $n$ ,  $CB$  must be taken a mean proportional between  $Cn$  and  $CS$ .

*Proof.* The only point in the construction requiring proof is that  $SE$  cuts  $VW$  in its point of contact.

Now the chords of contact  $PQ$ ,  $PR$ ,  $RQ$  of the given tangents are parallel respectively to  $TE$ ,  $ET_1$ ,  $T_1T$ , which is impossible unless  $EP$  passes through  $S$  the intersection of  $TQ$  and  $TR$ .

**PROBLEM 72.** To describe an ellipse, the centre  $C$ , two points ( $A$  and  $B$ ) of the curve and a tangent  $Tt$  being given (Fig. 71).



[A second tangent can at once be drawn parallel to  $Tt$  on the opposite side of  $C$ , and at the same distance from it;  $A$  and  $B$  must lie between these lines.]

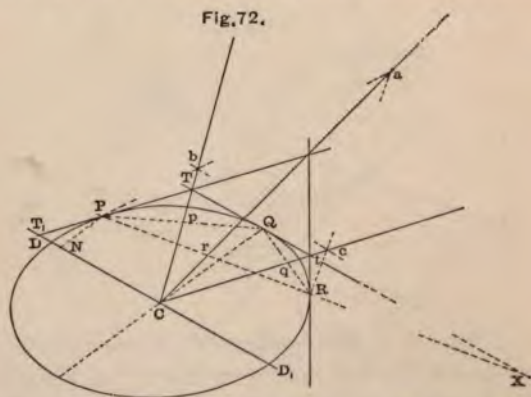
Through  $C$  draw  $CT$  parallel to  $AB$  meeting the given tangent in  $T$ . Bisect  $AB$  in  $N$  and draw  $NCt$  meeting  $Tt$  in  $t$ .  $CT$ ,  $Ct$  are a pair of conjugate diameters, and the problem reduces to Prob. 69. Draw the diameter  $ACA_1$  meeting  $Tt$  in  $L$ . Take  $Lm$  a mean proportional between  $LA$  and  $LA_1$ . Make  $LM$  on  $Lt$  equal to  $Lm$ , draw  $Mn$  perpendicular to  $Lt$  and equal to  $CA$  and  $Ln$  cutting a circle on  $Tt$  as diameter in  $p$  and  $p_1$ . Perpendiculars from  $p$  and  $p_1$  on  $Tt$  will determine two points, either of which can be taken as the point of contact of  $Tt$ , and the length  $pP$  will be the corresponding conjugate diameter  $CQ$ .

The construction is obvious from preceding problems.

PROBLEM 73. *To describe an ellipse, the centre  $C$  and three points  $P$ ,  $Q$ ,  $R$  being given (Fig. 72).*

[Any one of the three points, as  $R$ , must lie *between* one pair of the parallel lines furnished by the remaining points and their corresponding points on the other side of the centre, and outside the other pair.]

Bisect  $PQ$  in  $p$ ,  $QR$  in  $q$ , and  $RP$  in  $r$ , and draw  $Cp$ ,  $Cq$  and  $Cr$ , producing each indefinitely.  $PR$  is a double ordinate of the



on  $Cp$  and those at  $Q$  and  $R$  on  $Cq$ . If therefore a triangle be drawn the sides of which pass through  $P, Q, R$  and the vertices of which lie on  $Cp, Cq$ , and  $Cr$  respectively, the sides of this triangle will be the tangents at  $P, Q$  and  $R$ . This can be done by Prob. 15, p. 20:—Take *any point*  $a$  on  $Cr$ , draw  $Pa, Ra$  cutting  $Cp, Cq$  in  $b$  and  $c$  respectively; join  $bc$  cutting  $PR$  in  $X$ , and draw  $XQ$  cutting  $Cb$  in  $T$  and  $Cc$  in  $t$ :  $PT, Tt$ , and  $Rt$  will be the tangents at  $P, Q$ , and  $R$  respectively, and the problem may be completed by preceding problems, or thus; through  $C$  draw  $DCD_1$  parallel to  $Tt$  so that  $CD$  is conjugate to  $CQ$ ; let  $TP$  meet  $CD$  in  $T_1$ , draw  $PN$  parallel to  $CQ$  meeting  $CD$  in  $N$ . Take  $CD$  a mean proportional between  $CN$  and  $CT_1$ , and  $CD$  will be the extremity of the diameter  $CD$  (Prop. 3, p. 107).

The construction is obvious.

The given data are evidently equivalent to a diameter and two points of the curve.

**PROBLEM 74.** *To describe an ellipse, the foci  $F$  and  $F_1$  and a point  $Q$  on the curve being given* (Fig. 56).

It has been shewn already that the foci lie on the major axis and that  $FP + PF_1$  = the major axis (p. 101).

Bisect  $FF_1$  in  $C$ , and through  $C$  draw  $BCB_1$  perpendicular to  $FF_1$ . On  $CF, CF_1$  make  $CA = CA_1 = \frac{FQ + QF_1}{2}$ , and make  $FB = FB_1 = CA$ .  $AA_1, BB_1$  will be the axes of the required ellipse.

**PROBLEM 75.** *To describe an ellipse, the foci  $F$  and  $F_1$  and a tangent ( $PQ$ ) to the curve being given* (Fig. 67).

[ $PQ$  must not lie between  $F$  and  $F_1$ .]

From  $F$  draw  $FY$  perpendicular to  $PQ$  and produce it to  $L$  making  $YL = FY$ . Draw  $F_1L$  cutting  $PQ$  in  $Q$ , which will be the point of contact of  $PQ$  and the problem reduces to the preceding.

The construction is obvious from Prob. 68.

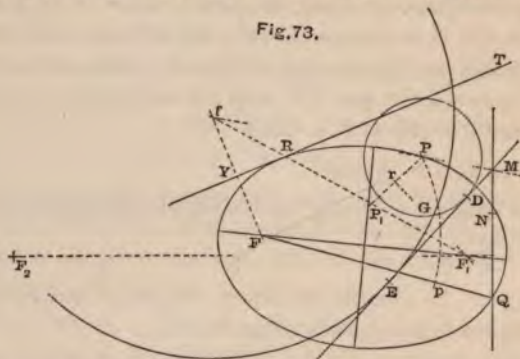


PROBLEM 76. To describe an ellipse, a focus  $F$ , a tangent  $RT$  with its point of contact  $R$ , and a second point  $P$  on the curve being given (Fig. 73).

From  $F$  draw  $FY$  perpendicular to  $RT$  meeting it in  $Y$ , and produce  $FY$  to  $f$  making  $Yf = YF$ .

[ $F$  and  $P$  must lie on the same side of  $RT$  and the distance of  $P$  from  $F$  must be less than its distance from a line drawn through  $f$  perpendicular to  $fR$ . See Problem 106, Chap. v.]

Fig. 73.



Draw  $fR$ , which will be a locus of the second focus. On  $fR$  towards  $R$  make  $fP_1 = FP$ . Draw  $PP_1$  and bisect it in  $r$ ; through  $r$  draw  $rF_1$  perpendicular to  $PP_1$  intersecting  $fR$  in  $F_1$ , which will be the second focus. Hence both foci being known the problem may be completed by Probs. 74 or 75.

*Proof.* That  $fR$  is a locus of the second focus has been shewn in Prob. 68; that the second focus lies on  $rF_1$  is evident thus: it must be so situated that

$$fR + RF_1 = FP + PF_1 = fF_1 = fP_1 + P_1F_1.$$

But  $FP = fP_1$ , therefore  $PF_1$  must be equal to  $P_1F_1$ , which by construction it is; therefore  $F_1$  is the second focus.

If  $fP_2$  be made =  $FP$  on  $Rf$  produced (i.e. on the side remote from  $R$ ), and a perpendicular to  $PP_2$  be drawn through the centre point of  $PP_2$  meeting  $Rf$  in  $F_2$ ,  $F$  and  $F_2$  will be the foci of an hyperbola fulfilling the given conditions.

**PROBLEM 77.** *To describe an ellipse, a focus  $F$ , a tangent  $RT$ , and two points  $P$  and  $Q$  of the curve being given (Fig. 73).*

[ $F$ ,  $P$ , and  $Q$  must all lie on the same side of  $RT$ .]

Let  $FQ$  be greater than  $FP$ , and on  $FQ$  make  $Fp = FP$ . With  $P$  as centre and radius  $= pQ$  describe a circle  $DG$ , then evidently the second focus must be equidistant from this circle and from the point  $Q$ , since the sum of the focal distances is constant. From  $F$  draw  $FY$  perpendicular to the given tangent  $RT$ , produce it to  $f$ , make  $Yf = YF$ , and with  $f$  as centre and radius  $FQ$  describe a circle  $EG$ : the second focus will evidently be equidistant from this circle and from the point  $Q$ , for it has been shewn (Prob. 68) that the distance of  $f$  from the second focus is equal to the major axis, and therefore equal to the sum of the focal distances of any point on the curve.

The problem therefore is reduced to finding the centre of a circle to touch externally two given circles ( $DG$ ,  $EG$ ) and pass through a given point ( $Q$ ), which is always possible since the circles must cut each other and  $Q$  lie outside both, i. e. the problem reduces to Prob. 32.

[Draw a common tangent  $EDM$  to the two circles meeting  $FP$  in  $M$ . Take  $MN$  on  $MQ$  such that

$$MN : MD :: ME : MQ,$$

and the second focus  $F_1$  will lie on the line perpendicular to  $NQ$  and passing through the centre point of  $NQ$ .]

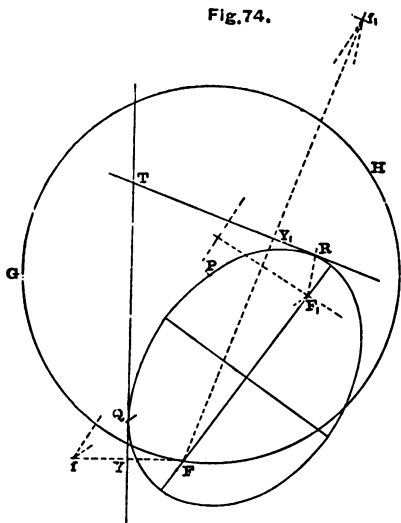
If the centre of the circle touching the above two circles *internally* be found (as  $F_2$ ),  $F$  and  $F_2$  will be the foci of an hyperbola which can be drawn through  $P$  and  $Q$  and touching  $RT$ . (See Prob. 107.)

**PROBLEM 78.** *To describe an ellipse, a focus  $F$ , a point  $P$  on the curve, and two tangents  $TQ$ ,  $TR$  being given (Fig. 74).*

[The points  $F$  and  $P$  must not lie on opposite sides of either tangent.]

From  $F$  draw  $FYf$  perpendicular to  $QT$  and  $FY_1f_1$  perpendicular to  $RT$ , meeting them respectively in  $Y$  and  $Y_1$ . Make  $Yf = YF$  and  $Y_1f_1 = Y_1F$ .

With centre  $P$  and radius  $PF$  describe the circle  $GH$ . Determine the centre ( $F_1$ ) of a circle to touch this circle internally



and to pass through  $f$  and  $f_1$  (Problem 27):  $F_1$  will be the second focus, and the axes can at once be determined by preceding problems.

*Proof.* It has been shewn (Prob. 68) that if  $F_1$  is the second focus,  $fF_1 = f_1F_1$  = the major axis  $= FP + PF_1$ , which by construction it does.

Referring to Problem 27 it will be seen that if the line  $ff_1$  cuts the circle  $GH$  and  $f$  and  $f_1$  lie on opposite sides of it a second ellipse can be drawn with foci  $F$  and  $F_1$ . If this second solution is impossible, a circle can generally be drawn passing through  $f$  and  $f_1$  and touching the circle  $GH$  externally.  $F$  and the centre of this circle will be the foci of a hyperbola fulfilling the conditions of the problem.

Hence either two ellipses or an ellipse and hyperbola can always be drawn to satisfy the given conditions.

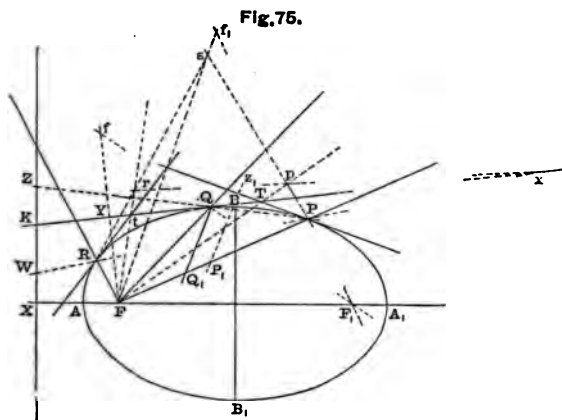
**PROBLEM 79.** *To describe an ellipse, a focus  $F$  and three tangents  $TP$ ,  $TQ$  and  $SR$  being given (Fig. 74).*

[The point  $F$  must lie within one of the three angles of the triangles (as  $PTQ$ ), i.e. it must not lie within either of the angles (as  $PSY_1$ ) where  $Y_1$  is on  $RS$  produced.]

From  $F$  drop perpendiculars  $FYf$ ,  $FY_1f_1$ ,  $FY_2f_2$  on the given tangents meeting them respectively in  $Y$ ,  $Y_1$ ,  $Y_2$ , and make  $Yf = YF$ ,  $Y_1f_1 = Y_1F$ ,  $Y_2f_2 = Y_2F$ ; then  $f, f_1, f_2$  must all be equidistant from the second focus (Prob. 68) and the problem therefore reduces to finding the centre ( $F_1$ ) of a circle which will pass through three given points. (Prob. 20.) To do this it is not really necessary to bisect  $ff_1$  and  $f_1f_2$  because it will be found that the perpendiculars through their points of bisection will pass through the points  $S$  and  $K$  in which the given tangents intersect, so that it is only necessary to draw through  $S$  and  $K$  perpendiculars to  $ff_1$  and  $f_1f_2$ , which will intersect in  $F_1$  the second focus. The major axis is of course known since it is equal to  $F_1f$ .

**PROBLEM 80.** *To describe an ellipse, a focus  $F$  and three points  $P$ ,  $Q$ ,  $R$  on the curve being given (Fig. 75).*

[The point  $F$  must lie within one of the three angles  $PQR$ ,  $QRP$ ,  $RPQ$ , and if circles be described with two of the given



points as centres passing through  $F$  and common tangents be drawn, the third point must be nearer to  $F$  than it is to the tangent more remote from  $F$ .]

*First Method.* It is a known proposition (Prop. 1, p. 105) that tangents drawn to an ellipse from any point subtend equal angles at the focus. The tangents at  $P$  and  $Q$  will therefore intersect on  $Fp$  the line bisecting the angle  $PFQ$ , those at  $R$  and  $Q$  will intersect on  $Fr$  the line bisecting the angle  $RFQ$ , and those at  $P$  and  $R$  will intersect on the line  $Fs$  bisecting the angle  $PFR$ . If therefore the three concurrent lines  $Fp$ ,  $Fr$ ,  $Fs$  be drawn and a triangle be constructed with its sides passing through  $P$ ,  $Q$  and  $R$  and with its vertices on the corresponding lines respectively (Prob. 15), these sides will be tangents to the curve at those points.

On  $Fs$  take any point  $s$ . Draw  $Rs$  cutting  $Fr$  in  $r$ , and  $Ps$  cutting  $Fp$  in  $p$ . Let  $RP$  and  $rp$  meet in  $x$ , and draw  $xQ$  cutting  $Fp$  in  $T$  and  $Fr$  in  $t$ .  $PT$ ,  $QT$ ,  $Rt$  will be the tangents at  $P$ ,  $Q$ ,  $R$  respectively, and the second focus can then be easily determined and the problem completed by preceding problems.

Although there are generally six solutions to Prob. 15, one only is available here, since the sides through the points have to terminate on definite pairs of lines.

*Second Method* (same fig.).

Draw  $FP$ ,  $FQ$  and  $FR$  and let  $FP$  be greater than  $FQ$  or  $FR$ .

Draw  $PQ$  and produce it to  $Z$  so that  $PZ : QZ :: FP : FQ$ ,

i. e. on  $FP$  make  $FP_1 = PQ$  and  $PQ_1 = FQ$ .

Through  $P_1$  draw  $P_1Z_1$  parallel to  $QQ_1$  meeting  $FQ$  in  $Z_1$ , and on  $PQ$  produced make  $QZ = FZ_1$ .  $Z$  will be a point on the directrix.

Similarly on  $PR$  produced take a point  $W$  such that

$$PW : RW :: FP : FR.$$

$W$  will be a second point on the directrix, which is therefore determined.

From  $F$  draw a perpendicular  $FX$  to  $WZ$  meeting it in  $X$ , and on  $FX$  take points  $AA_1$  such that  $FA : AX :: FA_1 : A_1X :: FP$  is to the perpendicular distance of  $P$  from  $XZ$ .  $AA_1$  will be the major axis.

The second focus and consequently the length of the major axis may perhaps be more easily determined thus. It is a known proposition (p. 102) that the tangent at any point, say  $Q$ , meets the directrix in a point  $K$  such that  $KFQ$  is a right angle. Therefore draw  $FK$  perpendicular to  $FQ$  meeting the directrix in  $K$ , and draw the tangent  $KQ$ . From  $F$  draw  $FYf$  perpendicular to  $KQ$  meeting it in  $Y$ , make  $Yf = YF$ , and draw  $fQ$  meeting  $XF$  in  $F_1$ , the second focus.  $fF_1$  is of course the length of the major axis.

*Proof.* Since by construction

$$PF : PZ :: QF : QZ,$$

therefore evidently  $PF$ : dist. of  $P$  from  $WZ$  ::  $QF$ : dist. of  $Q$  from  $WZ$ , and since  $PF : PW :: RF : RW$ ,

$$\therefore PF : \text{dist. of } P \text{ from } WZ :: RF : \text{dist. of } R \text{ from } WZ;$$

therefore the distances of the given points from the focus are in a constant ratio to their distances from  $WZ$ , which is therefore the directrix.

If the lines  $PQ$ ,  $PR$  are divided internally in the same ratio as above, two points are determined which being joined, either to each other or to the opposite points of the first pair, give three lines, either of which may be taken as the directrix of an *hyperbola* passing through the three given points and having  $F'$  as focus. In each case one of the given points will lie on one branch of the curve and two on the other.

Thus generally four conics can be drawn fulfilling the given conditions, one of which is an ellipse.

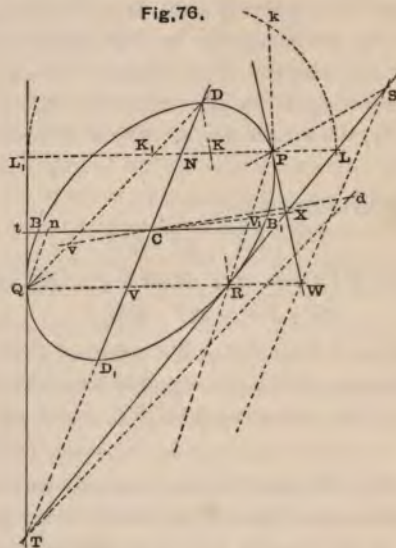
**PROBLEM 81.** *To describe an ellipse, two tangents  $TQ$ ,  $TR$  with their points of contact  $Q$  and  $R$ , and a point  $P$  on the curve being given (Fig. 76).*

[The point  $P$  must lie within the parabola which can be described touching  $TQ$ ,  $TR$  at  $Q$  and  $R$ .]

This is of course a simple case of the more general problem to describe an ellipse to touch two given lines and to pass through three given points.

*First Solution.* Let  $QP$  produced meet the tangent  $TR$  in  $S$ . From  $S$  draw a line passing through the intersection of  $PR$  and

Fig. 76.



the other tangent  $QT$  (Prob. 4) and meeting  $QR$  in  $W$ ; then  $W$  will be a point on the tangent at  $P$ , which can therefore be drawn. Let it intersect  $TR$  in  $X$ . Bisect  $QR$  in  $V$  and draw  $TVC$ , which will evidently be a diameter of the curve, i.e. is a locus of the centre. Bisect  $PR$  in  $V_1$  and draw  $XV_1C$ , which will similarly be a locus of the centre. The centre is therefore at  $C$ , the intersection of  $TV$  and  $XV_1$ , and the centre being known the problem can be completed by Probs. 70, 71, &c.

*Second Solution.* Bisect  $QR$  in  $V$  and through  $T$  draw  $TD, VD$ , which will evidently be a diameter of the ellipse, i.e. will pass through the centre. Through  $P$  draw  $LPNL_1$  parallel to  $QR$ , meeting  $TR$  in  $L$ ,  $TD$  in  $N$  and  $TQ$  in  $L_1$ . Take  $Pk$  a mean proportional between  $PL$  and  $PL_1$ , and from  $L$  and  $L_1$  towards  $N$  make  $LK = L_1K_1 = Pk$ ; then  $RK$  or  $QK_1$  will intersect  $TD$  in  $D$ ,



the extremity of the diameter. On  $TD$  take a point  $C$  such that

$$\begin{aligned} TC : CD :: CD : CV, \\ \text{i.e. } TC + CD : TC :: CV + CD : CD, \\ \text{or } TD : TC :: VD : CD. \end{aligned}$$

$C$  will be the centre of the ellipse.

[The point  $C$  can easily be found by drawing any two parallels through  $T$  and  $D$  (as  $Td$ ,  $Dv$ ), making  $Td = TD$  and  $Dv = VD$ , and joining  $dv$  cutting  $TD$  in  $C$ .]

The direction of the diameter  $CB$  conjugate to  $CD$  is known, since it is parallel to  $QR$ ; its length can easily be determined by taking a mean proportional between  $Cn$  and  $Ct$ , where  $n$  is the foot of the ordinate from  $Q$  on  $CB$  and  $t$  the intersection of  $CB$  and of the tangent at  $Q$ .

*Proof.* Let  $DM$  be the tangent at  $D$  meeting  $TR$  in  $M$ , and let  $LP$  meet the curve again in  $p$ , so that  $L_1P = Lp$ .

Then  $LP \cdot Lp : LR^2$  is the ratio of the squares of the parallel diameters (p. 117); but  $MD^2 : MR^2$  is the same ratio,

$$\begin{aligned} \therefore LP \cdot Lp : LR^2 :: MD^2 : MR^2 \\ \therefore LK^2 : LR^2 \text{ by similar triangles,} \\ \therefore LP \cdot Lp = LK^2, \text{ which justifies the construction.} \end{aligned}$$

**PROBLEM 82.** *To describe an ellipse, two tangents  $TP$ ,  $TQ$  and three points  $A$ ,  $B$ ,  $C$  on the curve being given (Fig. 77).*

[The points  $ABC$  must not lie on opposite sides of either line.]

Draw the line  $AB$  cutting the given tangents in  $P$  and  $Q$ . Find  $X$  the centre, and  $E$ ,  $E_1$ , the foci, of the involution  $A$ ,  $B$  and  $P$ ,  $Q$  (Prob. 13).

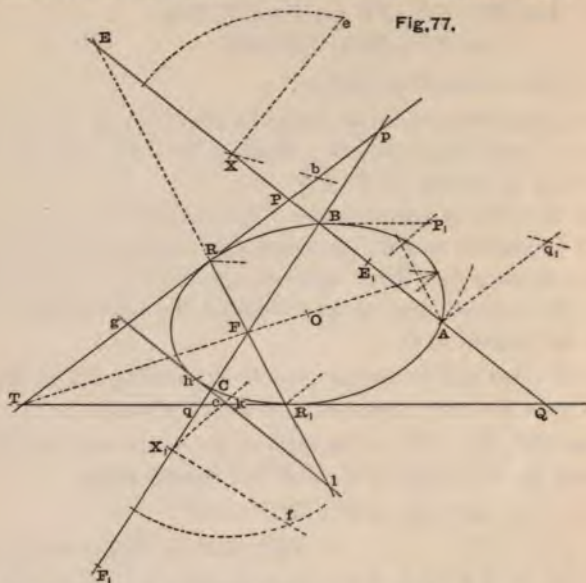
[In the figure,  $Pb$  on  $TP = PB$ ,  $Aq_1$  on a parallel to  $TP$  drawn through  $A$  is equal to  $AQ$ ; then  $q_1b$  cuts  $AB$  in  $X$ , the required centre.  $XE$  is a mean proportional between  $XA$  and  $XB$ .]

$E$  or  $E_1$  will be a point on the chord of contact of the given tangents.

Similarly draw  $BC$  cutting the given tangents in  $p$  and  $q$ , and find  $X_1$  the centre, and  $F$ ,  $F_1$  the foci of the involution  $B$ ,  $C$  and  $p$ ,  $q$ ; then  $F$  or  $F_1$  will be a second point on the chord of contact of the



given tangents, the points of contact of which  $R, R_1$  are therefore determined, and the problem reduces to the preceding\*. Since



$E$  and  $E_1$  can be joined to either  $F$  or  $F_1$  four chords of contact can in general be drawn, but one at least of the corresponding conics will be an hyperbola.

For proof that  $E$  and  $F$  are points on the chord of contact see Prop. 7, p. 143.

PROBLEM 83. To describe an ellipse, two points  $A, B$  on the curve, and three tangents  $PQ, QR, RP$  being given (Fig. 78).

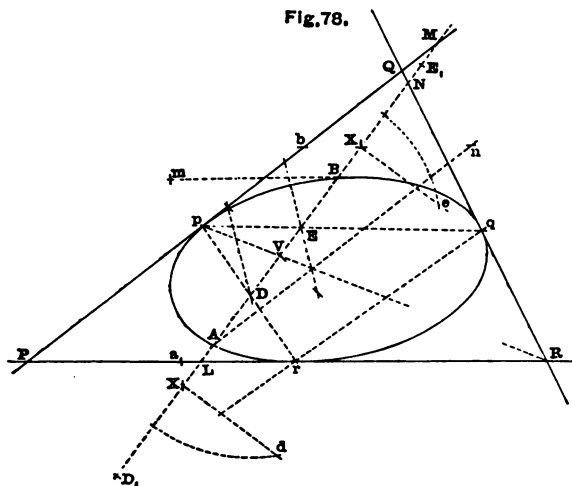
[ $A$  and  $B$  must not lie on opposite sides of either line.]

Draw a line through  $AB$  cutting the tangents through  $P$  in  $L$  and  $M$  and the remaining tangent in  $N$ .

Find  $X$  the centre, and  $D, D_1$  the foci of the involution  $AB$  and  $LM$  (Prob. 13).  $D$  or  $D_1$  will be a point on the chord of contact of the tangents  $PQ, PR$ .

\* In the figure the point of bisection of  $RR_1$  accidentally coincides with  $F$ .

[In the fig.  $La$  on  $PR=LA$ ,  $Bm$  on a parallel to  $PR$  drawn through  $B=BM$ , and  $ma$  cuts  $AB$  in  $X$ , the required centre.  $XD$  is a mean proportional between  $XA$  and  $XB$ .]



Similarly find  $X$ , the centre, and  $E, E_1$  the foci of the involution  $A, B$  and  $M, N$  (Prob. 13), and  $E$  or  $E_1$  will be a point on the chord of contact of the tangents  $QP, QR$ .

[In the fig.  $Mb$  on  $Qp = MB$ ,  $An$  on a parallel to  $QP$  through  $A$  is equal to  $AN$ , and  $bn$  cuts  $AB$  in  $X$ , the required centre.  $XE$  is a mean proportional between  $XM$  and  $XN$ .]

Find  $MV$ , the harmonic mean between  $ME$  and  $MD$ ,  $M$  being the point on the given tangents which has appeared in each of the above involutions (Prob. 11); then  $RV$  will cut the opposite tangent  $PQ$  in its point of contact ( $p$ ) with the curve, and therefore  $pEq$  will be the chord of contact of the tangents  $QP, QR$  and  $pDr$  that of  $PQ, PR$ . The problem therefore reduces to No. 81.

The construction depends on the property made use of in the last problem and proved in Prop. 7, p. 143, that the chord of contact of  $PQ, PR$  must pass through  $D$  or  $D_1$ , the foci of the involution  $AB$  and  $LM$ , and similarly that the chord of contact of  $QP$  and

$QR$  must pass through  $E$  or  $E_1$ , the foci of the involution  $AB$  and  $MN$ .

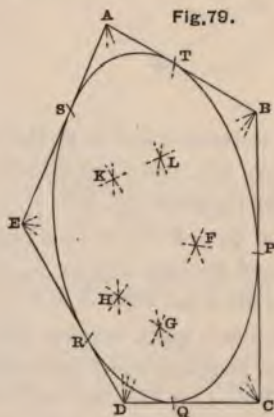
Also if  $rq$  meets  $VR$  in  $u$  and the tangent  $PQ$  in  $T$ , since  $DVEM$  is harmonic (by construction) so also is  $Truq$ , and therefore  $uV$  is the polar of  $T$  and therefore determines  $p$ , the point of contact of  $PV$  (Prop. 5, p. 141).

Since either  $D$  or  $D_1$  may be taken with  $E$  or  $E_1$ , there are in general four solutions.

PROBLEM 84. *To describe an ellipse to touch five given lines  $AB, BC, CD, DE, EA$ .*

[The lines must form a pentagon without a re-entering angle and the vertices are supposed to be lettered consecutively.]

Draw  $AC$  and  $BD$  intersecting in  $F$ . Then  $EF$  will intersect  $BC$  in  $P$ , the point of contact of  $BC$ . Similarly if  $BD$  and  $CE$



intersect in  $G$ ,  $AG$  will intersect  $CD$  in  $Q$ , the point of contact of  $CD$ ; and continuing the construction  $R, T$  and  $V$ , the points of contact of  $DE, EA$  and  $AB$  may be determined.

The centre of the curve can easily be found and the curve completed by preceding problems.

The construction depends on Brianchon's well-known theorem:

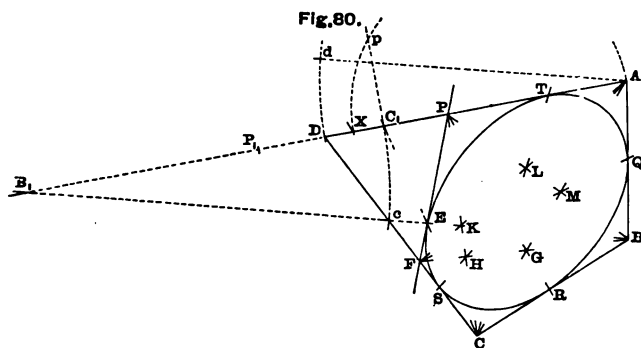
*"The three opposite diagonals of every hexagon circumscribing a conic intersect in a point."*

For if  $T$  be the point of contact of  $AB$  the pentagon may be considered as a hexagon  $AT, TB, BC, CD, DE, EA$ , and therefore  $AC, BE$  and  $DT$  must meet in a point  $L$ ; and conversely if  $L$  is the point of intersection of  $AC$  and  $BE$ ,  $DL$  must pass through  $T$ , the point of contact of  $AB$ , and similarly for the remaining sides.

**PROBLEM 85.** *To describe an ellipse, four tangents  $AB, BC, CD, DA$  and a point  $E$  on the curve being given (Fig. 80).*

[The point  $E$  must lie within the quadrilateral  $ABCD$ , which must not be a parallelogram.]

Let  $BE, CE$  meet  $AD$  in  $B_1$  and  $C_1$  respectively.



Find  $X$  the centre, and  $P$  and  $P_1$  the foci of the involution  $AC_1$  and  $DB_1$ . Prob. 13.

Then the tangent at  $E$  must pass through  $P$  or  $P_1$  and the problem reduces to the preceding.

There are two solutions.

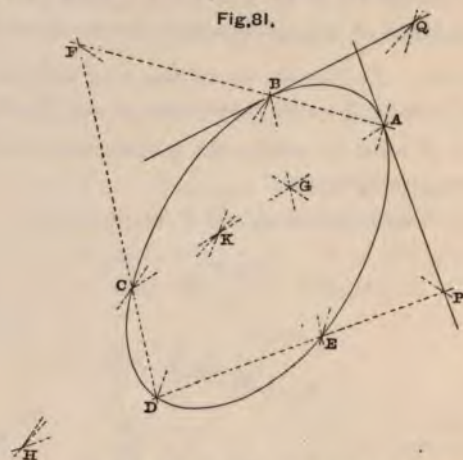
In the figure  $B_1c$  on  $B_1B = B_1C_1$ ;  $Ad$  on a parallel to  $B_1B$  is equal to  $AD$  and  $cd$  intersects  $AB_1$  in  $X$ , the required centre.  $XP$  is a mean proportional between  $XA$  and  $XC_1$ .

Also  $BP$  and  $AF$  intersect in  $L$  and  $CL$  will pass through  $T$ , the point of contact of  $AP$ .

PROBLEM 86. *To describe an ellipse to pass through five given points  $ABCDE$  (Fig. 81).*

[No point must lie inside the quadrilateral formed by the other four.]

Let  $AB, DC$  meet in  $F$  and  $AC, BE$  in  $G$ .



Draw  $FG$  meeting  $DE$  in  $P$ .  $P$  will be a point on the tangent at  $A$ .

Similarly if  $BC$  and  $ED$  meet in  $H$  and  $AC, BD$  in  $K$ ,  $HK$  will meet  $EA$  in  $Q$ , a point on the tangent at  $B$ .

The problem can evidently be completed in various ways by preceding problems.

The construction depends on Pascal's well-known theorem: "*The three intersections of the opposite sides of any hexagon inscribed in a conic section are in one right line.*" For the tangent at  $A$  may be considered as meeting the curve in two consecutive points  $A$  and  $a$ , and therefore  $P$ , the intersection of  $Aa$  and  $DE$ , must lie on  $FG$ , the straight line through the intersections of  $AB$  and  $DC$  and of  $BE$  and  $Ca$ .

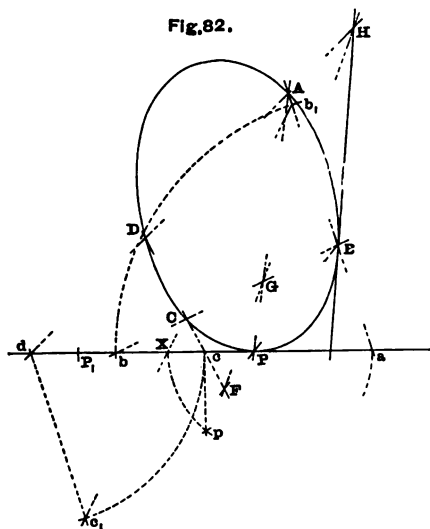
This line is known as the Pascal line.

There is only one solution.

**PROBLEM 87.** *To describe an ellipse, four points on the curve  $A, B, C, D$  and a tangent  $ad$  being given (Fig. 82).*

**[All the points must lie on the same side of the tangent.]**

Draw  $AB$  meeting  $ad$  in  $a$ ,  $BC$  meeting  $ad$  in  $b$ ,  $DC$  meeting it in  $c$ , and  $AD$  meeting it in  $d$ .



Find  $X$  the centre, and  $P$  and  $P_1$  the foci of the involution  $ac$  and  $bd$ .

$P$  or  $P_1$  will be the point of contact of the given tangent and the problem may be completed by several preceding ones.

In the fig.  $ab_1$  on  $aA=ab$ ;  $dc_1$  on a parallel to  $aA=dc$ , and  $b_1c_1$  intersects  $ad$  in  $X$ , the required centre.  $XP=Xp$ , a mean proportional between  $Xc$  and  $Xa$ .

If  $DC$ ,  $BP$  meet in  $F$ , and  $BC$ ,  $PA$  in  $G$ , then  $FG$  and  $DA$  will intersect in  $H$ , a point on the tangent at  $B$ .

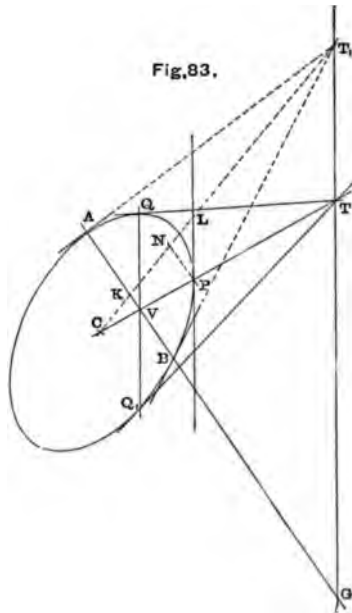
There are of course two solutions, as either  $P$  or  $P_1$  may be taken as the point of contact.

That  $P$ , the point of contact, is a focus of the involution is proved in Chapter 8.

## POLE AND POLAR.

It has been shewn in the case of the circle (Cor. 3, p. 31) that the pairs of tangents drawn at the extremities of any chord through a fixed point intersect in a straight line.

This is also true in the case of any conic section, for let  $V$  (fig. 83) be any point in a conic and  $C$  the centre, and let  $CV$



meet the curve in  $P$ . Take  $T$  in  $CV$  produced such that  $CV : CP :: CP : CT$ , and through  $V$  draw the chord  $QQ_1$  parallel to the tangent at  $P$ .

$QQ_1$  will be the chord of contact of the pair of tangents drawn from  $T$  to the conic, and will be bisected in  $V$ .

Through  $V$  draw any chord  $AVB$  and let the tangents at  $A$  and  $B$  intersect in  $T_1$ .

Join  $CT_1$ , and draw  $PN$  parallel to  $AB$ , meeting  $CT_1$  in  $N$ . Then if  $CT_1$  meet  $AB$  in  $K$  and the tangent at  $P$  in  $L$ ,

$$CK \cdot CT_1 = CN \cdot CL \quad (\text{Prop. 3, p. 107.})$$

$$\therefore CT_1 : CL :: CN : CK$$

$$:: CP : CV$$

$$:: CT : CP;$$

hence  $TT_1$  is parallel to  $PL$ , and therefore  $T_1$ , the intersection of the tangents at the extremities of *any* chord through  $V$ , lies on a fixed line.

DEF. As in the circle, the line  $TT_1$  is called the *polar* of the point  $V$  with respect to the conic and the point  $V$  is called the *pole* of  $TT_1$  with respect to the conic.

If the *pole* lies without the conic (as  $T$ ), its *polar* is the line  $QQ_1$  parallel to the tangent at the point ( $P$ ) where  $CT$  meets the conic, and meeting  $CT$  in a point  $V$  such that

$$CV : CP :: CP : CT,$$

i.e. is the chord of contact of tangents from the pole.

If the conic be a parabola, since the centre may be considered as at an infinite distance, the line  $VT$  must be drawn parallel to the axis meeting the curve in  $P$  and  $PT$  be made equal to  $PV$ , the *polar* of  $V$  will then be parallel to the tangent at  $P$  and will pass through  $T$ .

If the pole be on the curve, the polar is the tangent at the point.

The directrix is the polar of the corresponding focus.

If a point (as  $T_1$ ) lies on the polar of  $V$ , the polar of  $T_1$  passes through  $V$ .

The following important harmonic properties should be noticed.

PROP. 5. A straight line drawn through any point is divided harmonically by the point, the curve, and the polar of the point.





PROP. 6. If two tangents be drawn to a conic, any third tangent is harmonically divided by the two tangents, their chord of contact, and the point in which it touches the curve.

Let  $LMAN$  (fig. 84) be the third tangent meeting  $PQ$  in  $L$ , and  $TP$ ,  $TQ$  in  $M$  and  $N$ . Through  $N$  draw  $Nacb$  parallel to  $TP$  meeting the curve in  $a$ ,  $b$  and  $PQ$  in  $c$ .

Then  $Na . Nb : \overline{NQ}|^2 :: \overline{TP}|^2 : \overline{TQ}|^2$  (p. 117)

$$:: \overline{Nc}|^2 : \overline{NQ}|^2,$$

$$\therefore Na . Nb = \overline{Nc}|^2;$$

but  $LN^2 : \overline{LM}|^2 :: \overline{Nc}|^2 : \overline{PM}|^2$  by similar triangles,

$$\therefore \overline{LN}|^2 : \overline{LM}|^2 :: Na . Nb : \overline{PM}|^2$$

$$:: \overline{NA}|^2 : \overline{AM}|^2 \text{ (p. 117),}$$

i.e.  $LMAN$  is divided harmonically.

PROP. 7. If a straight line meet two tangents to a conic in  $PQ$  and the curve in  $AB$ , the chord of contact of the tangents will pass through one of the foci of the involution  $P$ ,  $Q$  and  $A$ ,  $B$  (fig. 77).

Since  $X$  is the centre and  $E, E_1$  the foci of the involution  $P, Q$  and  $A, B$ ,

$$XP : XA :: XB : XQ;$$

$$\therefore XA - XP : XA :: XQ - XB : XQ,$$

$$\text{or } PA : XA :: BQ : XQ.$$

Similarly  $PB : XB :: AQ : XQ,$

$$\therefore PA . PB : AQ . BQ :: XA . XB : \overline{XQ}|^2 :: \overline{XE}|^2 : \overline{XQ}|^2;$$

but (p. 18)  $EP : EQ :: PE_1 : E_1Q$ , since  $EPE_1Q$  is harmonic;

$$\therefore EP : EP + PE_1 :: EQ : EQ + QE_1,$$

$$\text{or } EP : EQ :: 2XE : 2XQ;$$

$$\therefore PA . PB : AQ . BQ :: \overline{EP}|^2 : \overline{EQ}|^2 \dots\dots\dots(1).$$

Draw the tangent  $ghkl$  parallel to  $PQ$  meeting  $TP, TQ$  in  $g$  and  $k$ , the chord of contact in  $l$ , and touching the curve in  $h$ ; and

if the chord of contact does not pass through  $E$  let it meet  $PQ$  in  $G$ .

$$PB \cdot PA : \overline{gh}^2 :: \overline{PR}^2 : \overline{Rg}^2 \text{ (p. 117)}$$

$$:: \overline{PG}^2 : \overline{lg}^2 \text{ by similar triangles,}$$

$$\text{and } QA \cdot QB : \overline{kh}^2 :: \overline{QR}^2 : \overline{kR}^2$$

$$:: \overline{QG}^2 : \overline{lk}^2;$$

but  $lkhg$  is harmonic (Prop. 6),

$$\text{and } \therefore \overline{lk}^2 : \overline{lg}^2 :: \overline{kh}^2 : \overline{gh}^2,$$

$$\therefore PB \cdot PA : QA \cdot QB :: \overline{GP}^2 : \overline{GQ}^2 \dots\dots\dots (2);$$

but (1) and (2) cannot be simultaneously true unless the points  $E$  and  $G$  coincide.

PROP. 8. If a quadrilateral be inscribed in a conic, its opposite sides and diagonals will intersect in three points such that each is the pole of the line joining the other two.

This follows at once from the harmonic properties of a complete quadrilateral, p. 16, combined with Prop. 5, p. 141. For since  $ECfA$  (fig. 11) is harmonic it follows that  $f$  is a point on the polar of  $E$  with respect to any conic passing through  $A$  and  $C$ , and since  $EDf_1B$  is harmonic  $f_1$  is a point on the polar of  $E$  with respect to any conic passing through  $BD$ . Therefore  $ff_1$ , i. e.  $OF$ , is the polar of  $E$  with respect to a conic passing through  $ABCD$ . Similarly  $OE$  is the polar of  $F$ . Also since  $O$  is on the polar of  $E$  the polar of  $O$  must pass through  $E$ , and since it is also on the polar of  $F$  the polar of  $O$  must pass through  $F$ , i. e.  $EF$  is the polar of  $O$ .

The triangle  $EFG$  is of course *self-conjugate* with respect to any conic circumscribing the quadrilateral, Def. p. 32.

PROP. 9. If a quadrilateral circumscribe a conic its three diagonals form a self-conjugate triangle (fig. 85a).

Let  $ABCD$  be the quadrilateral and let  $AB$  and  $CD$  intersect in  $G$ ,  $AC$  and  $BD$  in  $E$ , and  $AD$  and  $BC$  in  $F$ . Let  $BD$  and  $AC$  meet  $FG$  in  $K$  and  $L$  respectively. The triangle  $EKL$  is self-conjugate with respect to any conic inscribed in the quadrilateral  $ABCD$ .

Let the polar of  $F$  (i. e. the chord of contact  $PP_1$ ) meet  $FG$  in  $R$ ; then, since  $R$  is on the polar of  $F$ , it follows that  $F$  is on the polar of  $R$ .

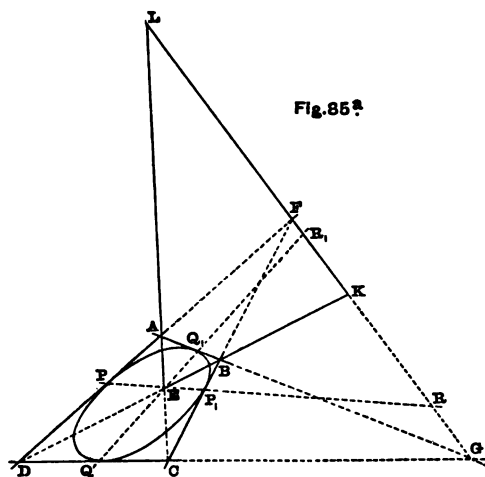


Fig. 85.

Now  $F(AEBG)$  is a harmonic pencil (p. 16), and if  $PP_1$  does not pass through  $E$  let  $FE$  meet  $PP_1$  in  $T$ ; then  $PTP_1R$  is a harmonic range; hence by (Prop. 5, p. 141)  $FE$  is the polar of  $R$ .

Similarly, if the other chord of contact  $QQ_1$  meet  $FG$  in  $R_1$ ,  $GE$  is the polar of  $R_1$ .

$\therefore E$  is the pole of  $RR_1$ , i. e. of  $LK$ .

Again,  $DEBK$  is a harmonic range, and if  $QP$  meet  $AC$  in  $S$  and  $CK$  in  $V$ ,  $QSPV$  is harmonic, and therefore  $S$  is on the polar of  $V$ ; but  $S$  is also on the polar of  $C$ , therefore  $CV$  or  $CK$  is the polar of  $S$ . Similarly, if  $P_1Q_1$  meet  $AC$  in  $S_1$ ,  $AK$  is the polar of  $S_1$ .

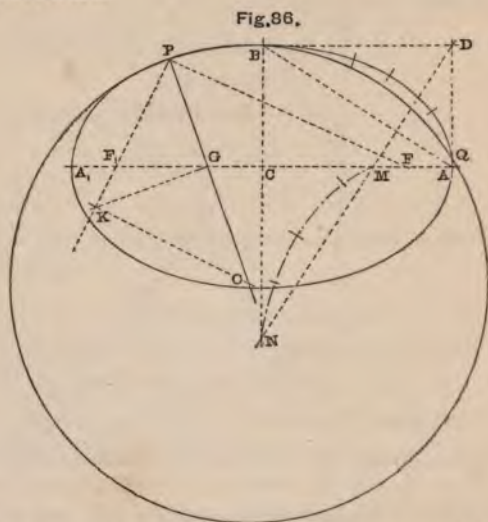
$\therefore K$  is the pole of  $LS_1$ , i. e. of  $EL$ ;

$\therefore ELK$  is a self-conjugate triangle.

**PROBLEM 88.** To determine the centre of curvature at any point  $P$  of a given ellipse (see page 89), fig. 86.

$CA, CB$  are the semi-axes, and  $F, F_1$  the foci. Draw  $PG$  the normal at  $P$  meeting the major axis in  $G$ , and draw  $GK$  perpen-

dicular to  $PG$  meeting  $PF$  or  $PF_1$ , the focal radii through  $P$  in  $K$ .  $KO$  perpendicular to  $PK$  will intersect  $PG$  in  $O$ , the required centre of curvature.



If  $AD$ ,  $BD$  be drawn parallel to the axes and  $DN$  be drawn perpendicular to  $AB$  meeting the major axis in  $M$  and the minor in  $N$ ,  $M$  and  $N$  will be the centres of curvature at  $A$  and  $B$  respectively. The evolute of the quadrant  $AB$  will therefore touch the axes at these points, and the evolute of the entire ellipse is made up of four curves similar to the chain dotted curve shewn in the figure between  $M$  and  $N$ .

As in the parabola, if the circle of curvature at  $P$  cuts the curve again in  $Q$ ,  $PQ$  is inclined to the axes at the same angles as is the tangent at  $P$ .

The construction depends on the known value of the radius of curvature,  $\frac{PG}{\cos^2 F_1 PG}$  (Salmon's *Conic Sections*, Chap. XIII.),

$$\text{for } \cos F_1 PG = \frac{PK}{PO} = \frac{PG}{PK},$$

and therefore rad. of curvature =  $PO$ .

EXAMPLES ON CHAPTER IV.

1. Describe an ellipse to touch a given straight line ( $QY$ ) and pass through a given point ( $P$ ); a focus  $F$  and the length ( $2a$ ) of the major axis being given.

[From  $F$  draw  $FY$  perpendicular to  $QY$  and produce it to  $T$ , making  $YT = YF$ . With  $T$  as centre and  $2a$  as radius describe an arc, and with  $P$  as centre and  $(2a - FP)$  as radius describe a second arc intersecting the former in  $F_1$ , which will be the second focus. There are two solutions.]

2. Describe an ellipse to touch two given straight lines; a focus and the length of the major axis being given (last question).

3. Describe an ellipse to touch two given lines  $OP$ ,  $OQ$  at the points  $P$  and  $Q$ ; one focus ( $F$ ) being on the line  $PQ$  and the angle  $POQ$  less than a right angle.

[The second focus  $F_1$  is the point of intersection of lines making with the given tangents angles equal to  $OPQ$ ,  $OQP$  respectively, i. e.  $OPF_1 = \pi - OPQ$  and  $OQF_1 = \pi - OQP$ . Bisect  $PQ$  in  $V$ ; centre lies on  $OV$ . Draw  $F_1K$  parallel to  $PQ$  meeting  $OV$  in  $K$ . Bisect  $VK$  in  $C$ , which will be the centre of required ellipse.]

4. Given one focus  $F$  of an ellipse, the length  $2b$  of the minor axis, and a point  $P$  on the curve; draw the locus of the centre.

[A parabola with the centre point of  $FP$  as focus,  $FP$  as axis and latus rectum  $= 2 \frac{b^2}{FP}$ .]

5. Given one focus  $F$  of an ellipse, the length  $2b$  of the minor axis and a tangent to the curve; shew that the locus of the second focus is a straight line parallel to the given tangent and at a distance from it  $= \frac{b^2}{p}$ , where  $p$  is the perpendicular from  $F$  on the given tangent.



6. Any focal radius  $FP$  is drawn in an ellipse, and the point  $Q$  on the auxiliary circle corresponding to  $P$  is joined to the centre  $C$ . Shew that the locus of the intersection of  $FP$  and  $CQ$  is an ellipse having  $F$  and  $C$  for foci.

7. Given the base, and sum of sides of a triangle; shew that the locus of centre of inscribed circle is an ellipse, having given base as major axis.

8.  $AB$  and  $BC$  are two equal rulers of length  $a$ , jointed at  $B$ .  $P$  is a point on  $BC$  distant  $b$  from  $B$ . The end  $A$  of one ruler is fixed and the end  $C$  of the other moves along a right line  $AC$  through  $A$ . Shew that the locus of  $P$  is an ellipse with semi-axes  $a + b$  and  $a - b$ .

9. An ellipse slides between two lines at right angles to each other; shew that the locus of its centre is a circle of radius  $\sqrt{a^2 + b^2}$ , where  $a$  and  $b$  are the semi-axes of the ellipse.

10. Draw an ellipse, and from any point  $P$  on it draw lines  $PD$ ,  $PE$  equally inclined to the major axis and meeting the curve again in  $D$  and  $E$ ; draw  $PC$  perpendicular to the major axis meeting  $DE$  in  $C$ . If the tangent at  $P$  meet  $DE$  in  $O$ , shew that the triangle  $POC$  is isosceles.

11. Shew by construction that the normal  $PG$  at any point of an ellipse is an harmonic mean between the focal perpendiculars on the tangent at  $P$ .

12. Given one focus  $F$  of an ellipse, the length  $2b$  of the minor axis, and a point  $P$  on the curve; draw the locus of the other focus.

[A parabola with focus  $P$ , axis  $FP$  and latus rectum  $= 4 \frac{b^2}{FP}$ .]

13. Shew that the locus of intersection of tangents at the ends of conjugate diameters of a given ellipse (semi-axes  $a$  and  $b$ ) is an ellipse, the axes of which coincide in direction with the given ellipse, and the semi-lengths of which are  $\sqrt{2}a$  and  $\sqrt{2}b$ .

14.  $AB$  is a line cutting in  $A$  and  $B$  a circle, centre  $C$ ;  $Q$  is a point on the perpendicular from  $C$  on  $AB$  on the same side of  $AB$  as  $C$  and outside the circle. Shew that the locus of the point  $P$  moving so that the tangent from  $P$  to the circle is in a constant ratio to the distance of  $P$  from  $AB$  ( $Q$  being a point on the locus) is an ellipse touching the circle at  $A$  and  $B$ .

15. Given any point  $P$  on an ellipse, inscribe in the ellipse a triangle  $PQR$ , the bisectors of the sides of which shall pass through the centre.

[Take  $p$  the point on the auxiliary circle corresponding to  $P$ . In the circle inscribe the equilateral triangle  $pqr$ ; the points corresponding to  $q$  and  $r$  will be the vertices of the required triangle.]

16. Given two tangents  $TP$ ,  $TQ$ ; their points of contact  $P$  and  $Q$  and the radius of curvature ( $\rho$ ) at one of them ( $P$  suppose) describe the ellipse.

[ $TC$  bisecting  $PQ$  is a locus of the centre. Draw the circle circumscribing the triangle  $TPQ$  and let  $d$  be its diameter. Draw a straight line through  $P$  such that  $p$  (the perpendicular distance of any point on it from  $PT$ ) :  $q$  (the perpendicular distance of the same point from  $QP$ ) ::  $PT \cdot d$  :  $QT \cdot \rho$ , i.e. determine the ratio  $\frac{p}{q} = \frac{PT \cdot d}{QT \cdot \rho}$  (p. 10). This line is a second locus of the centre, which is therefore known.]

17. Draw an ellipse, a focus  $F$ , a tangent  $PT$ , its point of contact  $P$  and the radius of curvature ( $\rho$ ) at  $P$ , being given.

[Reverse the construction of Prob. 88 to determine  $G$ , the foot of the normal at  $P$ , and consequently the direction of the major axis.]

18. If  $P$  is any point on an ellipse and the ordinate  $Pp$  perpendicular to the major axis meets the auxiliary circle in  $p$ , the angle between the major axis and the radius of the circle through  $p$  is called the *eccentric angle* of  $P$ .



Shew that if  $P$  be any point on an ellipse, the eccentric angle of which is  $\alpha$ , three points  $A$ ,  $B$ , and  $C$  on the curve, the eccentric angles of which are  $-\frac{\alpha}{3}$ ,  $-\frac{\alpha}{3} + 120^\circ$  and  $-\frac{\alpha}{3} + 240^\circ$ , are such that the circle of curvature at each passes through  $P$ ; and verify that a circle can be described through  $A$ ,  $B$ ,  $C$ , and  $P$ , and that the bisectors of the sides of the triangle  $ABC$  pass through the centre of the ellipse.

19. Draw in a given ellipse a pair of conjugate diameters making a given angle with each other.

[On any diameter of the ellipse describe a segment of a circle containing the given angle (Prob. 30). If the points where the circle meets the ellipse be joined to the ends of the chosen diameters, the required conjugate diameters will be parallel to these chords. The least possible angle between conjugate diameters of a given ellipse is the angle between the diagonals of the rectangle formed by the axes.]

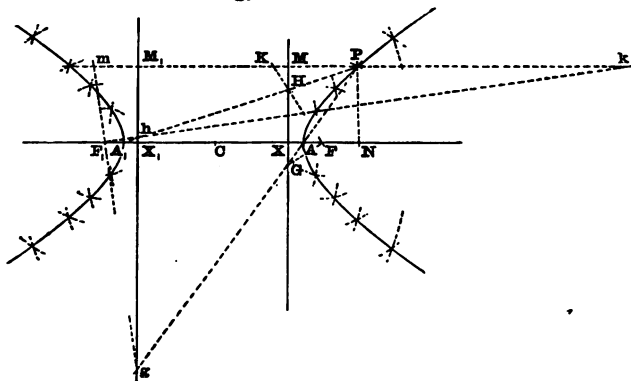
## CHAPTER V.

### THE HYPERBOLA.

As in the case of the ellipse, the definition of the curve given on page 56 does not immediately exhibit the property of the curve which furnishes the most convenient method of constructing it. It may also be defined as the locus of a point which moves in a plane, so that the difference of its distances from two fixed points in the plane is constant, and that the two definitions are really identical may be shewn thus:—

In fig. 87 let  $F$  be the focus and  $MX$  the directrix (Definitions, page 56).

Fig. 87.



From  $F$  draw  $FXF_1$  perpendicular to  $MX$  meeting it in  $X$ , and let  $A, A_1$  be points on  $FX$  such that  $\frac{FA}{AX} = \frac{FA_1}{A_1X} =$  the given

constant ratio (greater than unity) for all points on the curve: the points  $A$  and  $A_1$  are called the vertices of the curve, and  $AA_1$  the *transverse axis*, and if  $AA_1$  be bisected in  $C$ ,  $C$  is the *centre* of the hyperbola.

To shew that the curve can be constructed from a second focus and directrix corresponding to the vertex  $A_1$ .

Let  $P$  be a point on the curve, i.e. let

$$FP : PM :: FA : AX,$$

where  $PM$  is the perpendicular from  $P$  on the directrix.

Draw  $AP$ ,  $A_1P$  meeting the directrix in  $G$  and  $H$ , and let  $FH$  meet  $PM$  in  $K$ .

$$\begin{aligned} \text{Then} \quad PK : FA_1 &:: PH : A_1H \\ &:: PM : A_1X, \end{aligned}$$

$$\text{or} \quad PK : PM :: FA_1 : A_1X :: FA : AX;$$

$$\begin{aligned} \therefore PK = FP \text{ and the angle } PKF &= \text{the angle } PFK \\ &= \text{the angle } KFA_1. \end{aligned}$$

Similarly  $FG$  bisects the angle between  $FA_1$  and  $PF$  produced, therefore the angle  $HFG$  is a right angle.

In  $AA_1$  take a point  $X_1$  such that  $A_1X_1 = AX$ , and through  $X_1$  draw a straight line perpendicular to  $AA_1$ , and in  $FA_1$  produced take a point  $F_1$  such that  $A_1F_1 = AF$ .

Let  $PA_1$  and  $PA$  produced meet the perpendicular through  $X_1$  in  $h$  and  $g$  and join  $F_1g$ ,  $F_1h$ ,

$$\begin{aligned} \text{then} \quad gX_1 : GX &:: AX_1 : AX \\ &:: A_1X : A_1X_1 \\ &:: HX : hX_1, \end{aligned}$$

$$\therefore gX_1 \cdot hX_1 = GX \cdot XH = FX^2 = F_1X_1^2;$$

$$\therefore gF_1h \text{ is a right angle.}$$

Let  $PK$  (parallel to axis) meet  $gX_1$  in  $M_1$ ,  $gF_1$  in  $m$ , and  $F_1h$  produced in  $k$ ,

$$Pm : PM_1 :: F_1A : AX_1,$$

$$\text{and} \quad Pk : PM_1 :: F_1A_1 : A_1X_1,$$

$$\therefore Pm = Pk;$$

and  $mF_1k$  being a right angle,

$$F_1P = Pm = Pk,$$

$$\therefore F_1P : PM_1 :: F_1A_1 : A_1X_1,$$

and the curve can therefore be described by means of the focus  $F_1$  and the directrix  $X_1M_1$ .

It follows that the curve is symmetrical with regard to the centre  $C$ , and that it lies wholly without the tangents at the vertices  $A$  and  $A_1$ , which are perpendicular to  $CA$ .

We have at any point  $P$  of the hyperbola,

$$FP : PM :: FA : AX,$$

$$F_1P : PM_1 :: F_1A_1 : A_1X_1 \\ :: F_1A : AX_1;$$

$$\therefore F_1P - FP : PM_1 - PM :: F_1A - FA : AX_1 - AX;$$

but

$$PM_1 - PM = MM_1 = XX_1 = AX_1 - AX,$$

$$\therefore F_1P - FP = F_1A - FA = AA_1,$$

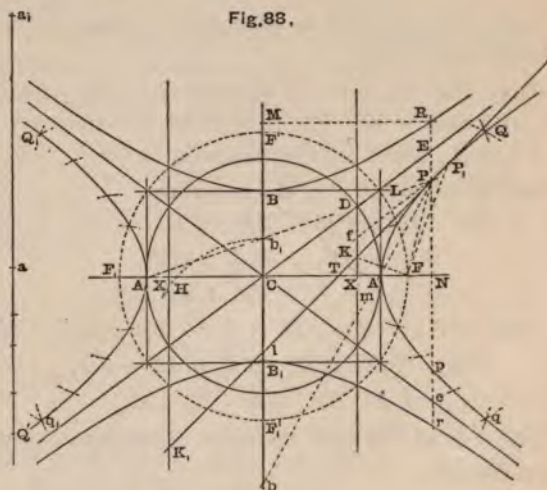
i.e. the difference of the focal distances is constant and equal to the transverse axis.

**PROBLEM 89.** *To describe an hyperbola, the foci and a vertex, or the vertices and a focus, or the transverse and conjugate axes being given (Fig. 88).*

Bisect the distance between the given foci  $F, F_1$  or the given vertices  $A, A_1$  in  $C$ .

With centre  $F$  and any radius greater than  $FA$  describe arcs as at  $Q$  and  $q$ , and with centre  $F_1$  and the same radius describe arcs as at  $Q_1$  and  $q_1$ . On any convenient line on the paper mark off a length  $aa_1 = AA_1$ , and with centre  $a$  and radius  $FQ$  mark off a point on this line on the opposite side from  $a_1$  as at  $Q'$ . Take off the distance  $Q'a_1$  from this line with a pair of dividers or compasses, and with centres  $F_1$  and  $F$  mark off points on the arcs already described about the opposite foci as centres. These points will of course be on the curve, since the difference of the focal distances of each is equal to  $AA_1$ , and the process may be repeated and as many points obtained as is necessary to define the curve and allow it to be sketched through the points with accuracy.

Though somewhat tedious, it is the only method for constructing the hyperbola which can be recommended.



Since the radii of the intersecting arcs may increase indefinitely, the curve evidently tends to infinity in both directions from  $C$ .

Through  $C$  draw  $BCB_1$  perpendicular to  $AA_1$ , and let the circle described on  $FF_1$  as diameter intersect the tangent at the vertex in  $L$ . Make  $CB = CB_1 = AL$ , then  $BB_1$  is called the *conjugate axis*; and if a second hyperbola be described with vertices at  $B$  and  $B_1$  and with foci on  $BB_1$  at  $F'$ ,  $F'_1$  the same distance from  $C$  as those of the original hyperbola, each curve is said to be conjugate to the other.

The eccentricity of the hyperbola (p. 57) is the numerical value of the ratio  $\frac{FA}{AX}$ . It is usually denoted by  $e$ , and its

value in terms of the axes is  $e = \frac{\sqrt{a^2 + b^2}}{a}$ ,

where

$$CA = a \text{ and } CB = b.$$

$$\text{For} \quad \frac{FA}{AX} = \frac{FA_1}{A_1X} = \frac{CF}{CA} = \frac{CL}{CA} = \frac{\sqrt{a^2 + b^2}}{a}.$$

The diagonals of the rectangle formed by the tangents to the hyperbola and its conjugate at their vertices are called *asymptotes*. The axes therefore bisect the angles between the asymptotes.

In fig. 87 let  $PN$  drawn from any point  $P$  of the curve perpendicular to the transverse axis meet it in  $N$ , and, as before, let  $PA$  and  $PA_1$  meet the directrix in  $G$  and  $H$ ,

$$\begin{aligned} \text{then} \quad & PN : AN :: GX : AX, \\ \text{and} \quad & PN : A_1N :: HX : A_1X; \\ \therefore & PN^2 : AN \cdot A_1N :: GX \cdot HX : AX \cdot A_1X \\ & :: FX^2 : AX \cdot A_1X, \end{aligned}$$

since  $GFH$  is a right angle,

i.e.  $\frac{PN^2}{AN \cdot A_1N}$  is a constant ratio.

$$\text{Since} \quad FA : AX :: FA_1 : A_1X,$$

$$\therefore FA + FA_1 : FA :: AX + A_1X : AX,$$

$$\text{or} \quad CF : CA :: FA : AX \dots\dots\dots(1),$$

$$\text{and} \quad FA_1 - FA : FA :: A_1X - AX : AX,$$

$$\text{or} \quad CA : CX :: FA : AX \dots\dots\dots(2),$$

$$\therefore CF : CA :: CA : CX :: FA : AX \dots\dots\dots(3).$$

$$\begin{aligned} \text{Also} \quad & CF : CX :: CF^2 : CF \cdot CX \\ & :: CF^2 : CA^2 \dots\dots\dots(4). \end{aligned}$$

Let the directrix meet the asymptotes in  $D$  (fig. 88): then by the similar triangles  $CDX$ ,  $CLA$ ,

$$CL : CA :: CD : CX;$$

but  $CL = CF$ , therefore from (3)  $CD = CA$ , or the circle on  $AA_1$  as diameter will cut the asymptote in a point on the directrix.

**DEF.** The circle on  $AA_1$  as diameter is called the auxiliary circle.

Since  $CD = CA$ ,  $CF = CL$ , and the angle  $DCF$  is common to the two triangles  $DCF$  and  $ACL$ ,

$\therefore$  the angle  $CDF =$  the angle  $CAL =$  a right angle,

or the perpendicular from the focus on the asymptote is a tangent to the auxiliary circle at the point of intersection.

COROLLARY.  $DX^2 = CX \cdot FX$ .

Again, from (4),

$$CF^2 - CA^2 : CA^2 :: CF - CX : CX \\ :: FX^2 : CX (CF - CX);$$

but  $CF \cdot CX = CA^2$  from (3), and  $CA^2 - CX^2 = AX \cdot A_1X$ .

Also

$$CF^2 - CA^2 = CB^2,$$

$$\therefore CB^2 : CA^2 :: FX^2 : AX \cdot A_1X;$$

and comparing this with the constant ratio above given for

$\frac{PN^2}{AN \cdot NA_1}$ , we have  $PN^2 : AN \cdot NA_1 :: BC^2 : AC^2$ , which may also be written

$$PN^2 : CN^2 - AC^2 :: BC^2 : AC^2 \dots\dots(5).$$

Let  $PN$  (fig. 88), where  $P$  is any point on the curve and  $PN$  the ordinate, meet the asymptotes in  $E$ , then

$$EN^2 : CN^2 :: BC^2 : AC^2, \text{ by similar triangles } \dots\dots(6),$$

$$\therefore EN^2 - PN^2 : AC^2 :: BC^2 : AC^2,$$

or

$$Ep \cdot EP = BC^2 = EP \cdot Pe,$$

where  $p$  is the point in which  $PN$  meets the curve again, and  $e$  is the point in which it meets the other asymptote.

Let the ordinate through  $P$  meet the conjugate hyperbola in  $R$  (same fig.), and let  $RM$  be the ordinate of  $R$  perpendicular to  $BC$ , then of course

$$RM^2 : CM^2 - BC^2 :: AC^2 : BC^2,$$

or

$$CM^2 : BC^2 :: RM^2 + AC^2 : AC^2;$$

but

$$CM = RN \text{ and } RM = CN,$$

$$\therefore RN^2 : BC^2 :: CN^2 + AC^2 : AC^2 \dots\dots\dots(7);$$

and combining (6) and (7), we get

$$RN^2 - EN^2 = BC^2 = RE \cdot Er = RE \cdot Re,$$

where  $r$  is the point on the other branch of the conjugate hyperbola corresponding to  $R$ , and  $e$  is the point in which  $Rr$  meets the other asymptote.

*To draw a tangent and normal at any point of the curve (Fig. 88).*

Let  $P_1$  be a point on the curve adjacent to any point  $P$ , and let the chord  $PP_1$  meet the directrices in  $K$  and  $K_1$ . Draw  $KK'$  to the corresponding focus: then  $FP : FP_1 :: PK : P_1K$ , or  $PK$  bisects the exterior angle between  $PF$  and  $P_1F$  produced (Euc. VI. prop. A). Hence, exactly as in the case of the ellipse (p. 102), when  $P_1$  moves up to and coincides with  $P$ , so that the chord  $PP_1$  becomes the tangent at  $P$ , the line  $FK$  becomes perpendicular to the line  $FP$  drawn from the focus to the point of contact of the tangent. *The tangent at any point  $P$  of an hyperbola may therefore be drawn* by drawing a line from  $P$  to either focus, erecting a perpendicular to this line at the focus meeting the directrix, and drawing the tangent through this point and the proposed point of contact. It may also be drawn by making use of the known property that *it bisects the angle between the focal distances*. For in the two triangles  $PFK$ ,  $PF_1K_1$

$$FP : PK :: F_1P : PK_1,$$

and the angle  $PFK =$  the angle  $PF_1K_1$ , each being a right angle,

$$\therefore \text{the angle } FPK = F_1PK_1. \quad (\text{Euc. VI. 7.})$$

Hence the normal bisects the exterior angle between the focal distances.

**PROBLEM 90.** *To describe an hyperbola, an asymptote  $CD$ , a focus  $F$ , and a point  $P$  being given (Fig. 88).*

From  $F$  draw  $FD$  perpendicular to  $CD$ , then  $D$  will be a point on the directrix as has been previously proved. Through  $P$  draw  $Pf$  parallel to  $CD$  and make  $Pf = PF$ , then  $f$  will be a second point in the directrix, which is therefore determined. Draw  $CF$



perpendicular to  $Df$  meeting the given asymptote in  $C$ , which will evidently be the centre of the curve. Make  $CA$ ,  $CA_1$  on  $CF$  each equal  $CD$  and  $AA_1$  will be the transverse axis, and the curve is completely determined.

Since  $Pf$  can be measured on either side of  $P$  there are generally two solutions.

*Proof.* The only step in the construction which is not obvious is taking  $f$  as a point on the directrix. It can easily be shewn to hold in the hyperbola, for draw  $Am$  parallel to the asymptote meeting the directrix in  $m$ , then in the hyperbola

$$\begin{aligned} FP : FA &:: PM : AX \\ &:: Pf : Am, \end{aligned}$$

where  $PM$  is a perpendicular on the directrix.

But  $Am = DL = AF$ ,  $\therefore FP = Pf$ ;

and conversely, if  $Pf$  be made  $= PF$ ,  $f$  will be a point on the directrix.

PROBLEM 91. To describe an hyperbola, an asymptote  $CT$ , a tangent  $Tt$ , and a focus  $F$  being given (Fig. 89).

From  $F$  draw  $FD$  perpendicular to  $CT$  meeting it in  $D$ , and  $FY$  perpendicular to  $Tt$  meeting in  $Y$ . Then  $D$  and  $Y$  are points on the auxiliary circle. Bisect  $DY$  in  $K$  and draw  $KC$  perpendicular to  $DY$  meeting  $CD$  in  $C$ .  $C$  will be the centre of the curve,  $CF$  the direction of the transverse axis, and  $CD$  or  $CY$  its semi-length.

PROBLEM 92. To describe an hyperbola, an asymptote  $CD$ , a directrix  $DD_1$  and a point  $P$  being given (Fig. 89).

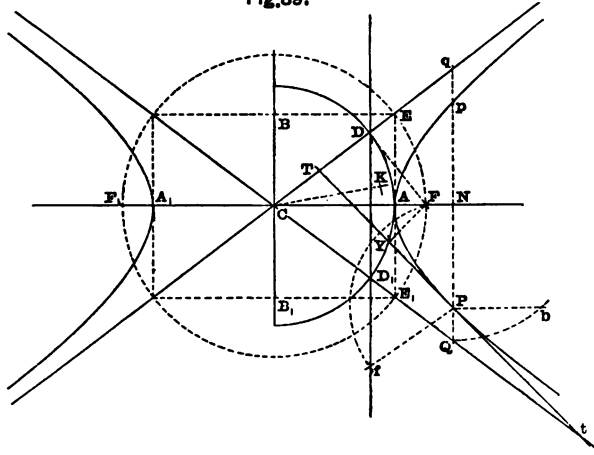
From  $D$  draw  $DF$  perpendicular to  $CD$ .  $DF$  will be a locus of the focus. Through  $P$  draw  $Pf$  parallel to  $CD$  meeting  $DD_1$  in  $f$ , and with centre  $P$  and radius  $Pf$  describe an arc cutting  $DF$  in  $F$ .  $F$  will be a focus, and  $FC$  drawn perpendicular to  $DD_1$  will intersect  $CD$  in  $C$ , the centre of the curve; which is therefore completely determined.

[The problem is exactly the converse of Prob. 90.]

**PROBLEM 93.** *To describe an hyperbola, the asymptotes  $CD, CD_1$  and a point  $P$  on the curve being given (Fig. 89).*

Bisect the angles between the asymptotes by the lines  $ACA_1, BCB_1$ ; then  $ACA_1$  in the angle in which  $P$  lies is the position of

Fig. 89.



the transverse axis. Through  $P$  draw  $QPq$  perpendicular to  $AA_1$ , and meeting the asymptotes in  $Q$  and  $q$ . Take a mean proportional, as  $Pb$ , between  $PQ$  and  $Pq$ .  $Pb$  will be the length  $CB$  of the conjugate semi-axis.

Draw  $BE$  parallel to  $AA_1$ , meeting the asymptote in  $E$ ; then  $BE$  is the length  $CA$  of the transverse semi-axis and  $CE = CF$ , the distance of either focus from  $C$ .

*Proof.* The only step in the construction requiring demonstration is that in the hyperbola  $BC^2 = PQ \cdot Pq$ .

Let  $N$  be the foot of the double ordinate  $Pp$ ; by similar triangles  $CAE, CNQ$ .

$$QN^2 : AE^2 :: CN^2 : AC^2 \text{ and } AE = BC,$$

$$\therefore QN^2 - BC^2 : BC^2 :: CN^2 - AC^2 : AC^2;$$

but  $CN^2 - AC^2 = AN \cdot NA_1$ , and (p. 156)  $PN^2 : AN \cdot NA_1 :: BC^2 : AC^2$ ,

$$\therefore QN^2 - BC^2 : BC^2 :: PN^2 : BC^2;$$

$$\therefore QN^2 - BC^2 = PN^2 \text{ or } QN^2 - PN^2 = BC^2,$$

$$\text{i.e. } (QN + PN)(QN - PN) = BC^2 = PQ \cdot Pq,$$

for by the symmetry of the curve  $Nq = NQ$ .

If  $qp$  be made equal to  $QP$ ,  $p$  will evidently be a point on the curve.

PROBLEM 94. To describe an hyperbola, the asymptotes  $CT$ ,  $Ct$ , and a tangent  $Tt$  to the curve being given (Fig. 89).

Bisect  $Tt$  in  $P$ .  $P$  will be the point of contact of  $Tt$ , i.e. will be a point on the curve, and the problem therefore reduces to Problem 93.

The proof will be found on p. 163.

DEFINITION. Any straight line drawn through the centre and terminated both ways either by the original curve or by the conjugate hyperbola is called a *diameter*, and by the symmetry of the curve every diameter is bisected by the centre. A diameter  $CD$  parallel to the tangent at the extremity of a diameter  $CP$  is said to be *conjugate* to  $CP$ .

The following important properties of the hyperbola should be carefully noticed.

PROP. 1. If from any point  $Q$  in an asymptote  $QPpq$  be drawn meeting the curve in  $P$ ,  $p$  and the other asymptote in  $q$ , and if  $CD$  be the semi-diameter parallel to  $Qq$ ,

$$QP \cdot Pq = CD^2 \text{ and } QP = pq \text{ (Fig. 90).}$$

Through  $P$  and  $D$  draw  $RP r$ ,  $DT t$  perpendicular to the transverse axis, and meeting the asymptotes in  $R$ ,  $r$  and  $T$ ,  $t$ ; let  $Rr$  meet the axis in  $N$ .

Then  $QP : RP :: CD : DT$  } by similar triangles,  
and  $Pq : Pr :: CD : Dt$  }

$$\therefore QP \cdot Pq : RP \cdot Pr :: CD^2 : DT \cdot Dt.$$

$$\text{But } RP \cdot Pr = BC^2 = DT \cdot Dt \text{ (p. 159),}$$

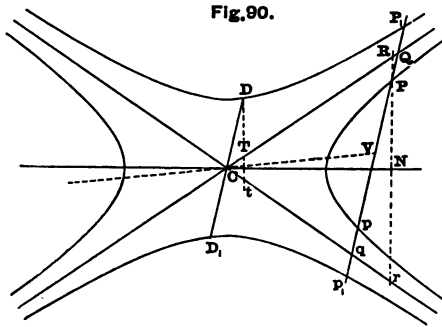
$$\therefore QP \cdot Pq = CD^2.$$

Similarly  $qp \cdot pQ = CD^{\circ} = QP \cdot Pq$ ;

or, if  $V$  be the middle point of  $Qq$ ,

$$QV^s - PV^s = QV^s - pV^s.$$

Hence  $PV = pV$ , and therefore  $PQ = pq$ .



**COR.** *If a straight line  $PP_1p_1p$  meet the hyperbola in  $P, p$ , and the conjugate hyperbola in  $P_1, p_1$ ,  $PP_1 = pp_1$ .*

For if the line meet the asymptote in  $Q, q$ ,

$$QP_1 = p_1q \text{ and } PQ = qp, \therefore PP_1 = pp_1.$$

**PROP. 2.** *A diameter bisects all chords parallel to the tangents at its extremities, i. e. all chords parallel to its conjugate.*

This can be proved exactly as in the analogous proposition for the ellipse.

Let  $QQ_1$  (fig. 91) be any chord of an hyperbola meeting the directrix in  $R$ , and let  $O$  be the centre point of  $QQ_1$  and  $F$  the focus.

Join  $FQ$ ,  $FQ_1$ , and draw  $FY$  perpendicular to  $QQ_1$ .

$$\begin{aligned} \text{Then } FQ^2 - FQ_1^2 &= QY^2 - Q_1Y^2 \\ &= (QY + Q_1Y)(QY - Q_1Y) \\ &= 2 \cdot QQ_1 \cdot OY \dots\dots\dots(1); \end{aligned}$$

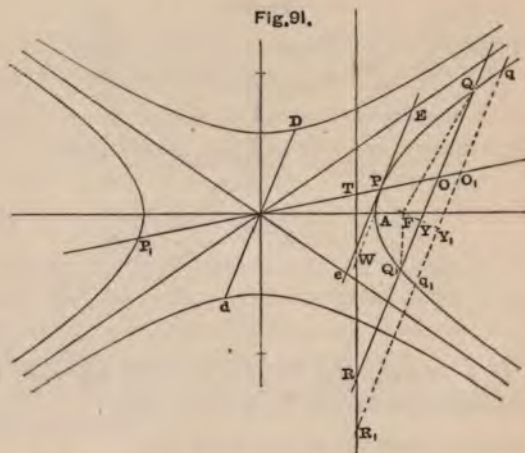
but since  $Q$  and  $Q_1$  are on the hyperbola,

$$FQ : FQ_1 :: QR : Q_1R;$$

therefore 
$$\frac{FQ^2 - FQ_1^2}{FQ^2} = \frac{QR^2 - Q_1R^2}{QR^2} = \frac{2QQ_1 \cdot OR}{QR^2} \dots\dots\dots(2);$$

therefore, from (1) and (2),

$$\frac{OY}{OR} = \frac{FQ^2}{QR^2} = \frac{FA^2}{AW^2},$$



where  $AW$  is drawn through the vertex parallel to  $QR$  meeting the directrix in  $W$ .

I.e.  $OY : OR$  in a constant ratio.

Take any second chord  $qq_1$  parallel to  $QQ_1$  meeting  $FY$  in  $Y_1$  and the directrix in  $R_1$ . Let  $O_1$  be its centre point; then, since  $\frac{OY}{OR} = \frac{O_1Y_1}{O_1R_1}$ , it follows that the line  $OO_1$  must pass through the point  $T$  in which  $FY$  meets the directrix, and is therefore fixed for all chords parallel to  $QQ_1$ . This line will evidently pass through the centre (i.e. will be a diameter), for by the last proposition it bisects all chords of the conjugate hyperbola parallel to  $QQ_1$ , i.e. it bisects the diameter  $Dd$ , which is also bisected by  $C$ .

Let  $TO$  meet the hyperbola in  $P$  and suppose  $qq_1$  to move parallel to itself till it approaches and ultimately coincides with  $P$ . Since  $O_1q = O_1q_1$  throughout the motion, the points  $q, q_1$  will evidently

approach  $P$  simultaneously, and in the limiting position  $qq_1$  will be the tangent at  $P$ . It follows that if  $P_1$  be the other extremity of the diameter through  $P$ , the tangent at  $P_1$  is parallel to  $QQ_1$ , and therefore to the tangent at  $P$ .

**COROLLARY 1.** The perpendicular on the tangent at any point from the focus meets the corresponding diameter in the directrix.

**COR. 2.** If the tangent at  $P$  meet the asymptotes in  $E$  and  $e$ ,  $PE = Pe$ , for by the last proposition the intercept between  $q$  and one asymptote is always equal to the intercept between  $q_1$  and the other asymptote, and when  $q$  and  $q_1$  ultimately coincide with  $P$  these intercepts become  $PE$  and  $Pe$  respectively, i.e. the portion of any tangent between the asymptotes is bisected at the point of contact.

**COR. 3.** If  $PE$  be the tangent at  $P$  meeting the asymptote in  $E$ ,  $PE^2 = CD^2$ , where  $CD$  is the semi-diameter conjugate to  $CP$ . For taking a parallel chord very near the tangent meeting the curve in  $p$ ,  $p_1$  and the asymptote in  $e$ , we have, by Prop. 1,

$$ep \cdot ep_1 = CD^2,$$

and therefore when  $p$  and  $p_1$  coincide in  $P$ ,

$$EP^2 = CD^2.$$

**COR. 4.** The asymptotes are the diagonals of the parallelogram formed by the tangents at the extremities of a pair of conjugate diameters. For  $E$  and  $e$ , which are on the asymptotes, are also angular points of such a parallelogram.

**PROP. 3.** *Tangents drawn at the extremities of any chord subtend equal angles at the focus.*

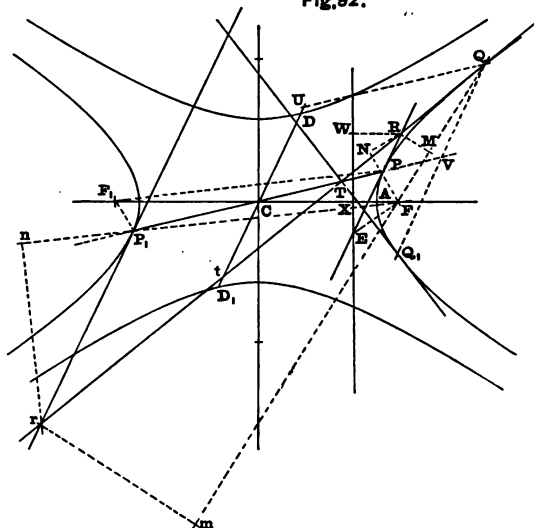
Let  $PQ$  (fig. 92) be any chord of an hyperbola and let the tangents at  $P$  and  $Q$  meet in  $R$ . Let  $F$  be the focus, and from  $R$  draw  $RN$ ,  $RM$  perpendicular respectively to  $FP$ ,  $FQ$ ; draw  $RW$  perpendicular to the directrix and let the tangent at  $P$  meet the directrix in  $E$ .

Then  $EF$  is perpendicular to  $FP$  (p. 157), and therefore parallel to  $RN$ .

Therefore  $FN : FP :: ER : EP$   
 $:: RW : PK,$

where  $PK$  is the perpendicular from  $P$  on the directrix.

Fig. 92.



Therefore  $FN : RW :: FP : PK$   
 $:: FA : AX.$

Similarly  $FM : RW :: FA : AX;$

therefore  $FN = FM.$

Hence in the right-angled triangles  $RFN$ ,  $RFM$ ,  $FN = FM$ , and  $FR$  is common.

Therefore the two triangles are equal in all respects, i.e. the angle  $RFP =$  the angle  $RFQ$ , and  $RN = RM$ .

PROP. 4. If  $PCP_1$  be a diameter and  $QVQ_1$  a chord parallel to the tangent at  $P$  and meeting  $PP_1$  produced in  $V$ , and if the tangent at  $Q$  meet  $PP_1$  in  $T$ , then  $CV \cdot CT = CP^2$  (Fig. 92).

Let  $TQ$  meet the tangents at  $P$  and  $P_1$  in  $R$  and  $r$ , and  $F$  being a focus draw  $RN$  perpendicular to the focal distance  $FP$  meeting

it in  $N$ ,  $rn$  perpendicular to  $FP_1$  meeting it in  $n$ , and  $RM$ ,  $rm$  perpendicular to the focal distance  $FQ$ . Let  $F_1$  be the other focus, and join  $F_1P$ ,  $F_1P_1$ .

Since  $CF = CF_1$ ,  $CP = CP_1$ , and the angle  $FCP =$  the angle  $F_1CP_1$ , therefore the triangles  $FCP$ ,  $F_1CP_1$  are equal in all respects; and therefore the angle  $CPF =$  the angle  $CP_1F_1$ .

Similarly the angle  $CPF_1 =$  the angle  $CP_1F$ .

Therefore the whole angle  $FPF_1 =$  the whole angle  $F_1P_1F$ ; but the tangents bisect the angles between the focal distances, therefore the angle  $FPR =$  the angle  $FP_1r$ ; i.e. the right-angled triangles  $RPN$ ,  $rP_1n$  are similar, and therefore

$$RP : rP_1 :: RN : rn;$$

but  $RN = RM$  and  $rn = rm$  (Prop. 3), therefore

$$RP : rP_1 :: RM : rm$$

$$:: RQ : rQ.$$

But

$$TR : Tr :: RP : rP_1$$

$$:: RQ : rQ;$$

therefore

$$TP : TP_1 :: PV : P_1V$$

by similar triangles,

$$\text{or } CP - CT : CT + CP :: CV - CP : CV + CP,$$

i.e.

$$CT : CP :: CP : CV;$$

therefore

$$CT \cdot CV = CP^2.$$

COR. 1. Since  $CV$  and  $CP$  are the same for the point  $Q_1$ , the tangent at  $Q_1$  passes through  $T$ , or the tangents at the extremities of any chord intersect on the diameter which bisects that chord.

PROP. 5. If  $PCP_1$ ,  $DCD_1$  be conjugate diameters, and  $QV$  be drawn parallel to  $CD$  meeting the hyperbola in  $Q$  and  $CP$  in  $V$ , then

$$QV^2 : PV \cdot P_1V :: CD^2 : CP^2.$$

Let the tangent at  $Q$  (fig. 92) meet  $CP$  and  $CD$  in  $T$  and  $t$  respectively, and draw  $QU$  parallel to  $CP$  meeting  $CD$  in  $U$ .

Then  $CV \cdot CT = CP^2$  and  $CU \cdot Ct = CD^2$  (Prop. 4);

but

$$CU = QV,$$

therefore

$$CD^2 : CP^2 :: QV \cdot Ct : CV \cdot CT;$$

but

$$Ct : QV :: CT : VT,$$

$$\therefore CD^2 : CP^2 :: QV^2 : CV \cdot VT,$$



and  $CV.VT = CV(CV - CT) = CV^2 - CP^2 = PV.P_1V$ ;

therefore  $QV^2 : PV.P_1V :: CD^2 : CP^2$ .

PROBLEM 95. *To describe an hyperbola, the transverse axis  $AA_1$  and a point  $P$  on the curve being given (Fig. 88).*

Bisect  $AA_1$  in  $C$ , which will of course be the centre of the curve. Draw the conjugate axis  $BCB_1$ . Let  $PA_1, PA$  cut  $BB_1$  in  $b_1$  and  $b$  respectively. Take a mean proportional  $CH$  between  $Cb$  and  $Cb_1$ , which will be the length ( $CB$  or  $CB_1$ ) of the semi-conjugate axis. The foci can then be determined, since  $CF = AB$ .

*Proof.* Let  $PN$  be the ordinate at  $P$ .

Then  $PN : b_1C :: NA_1 : CA_1$ ,

and  $PN : bC :: NA : CA$ ,

or  $PN^2 : NA.NA_1 :: bC.b_1C : CA^2$ ;

therefore  $bC.b_1C = BC^2$  (p. 156).

PROBLEM 96. *To describe an hyperbola, the transverse axis  $ATA_1$  and a tangent  $PT$  being given (Fig. 88).*

Bisect  $AA_1$  in  $C$ , the centre of the curve. On  $CA$  towards  $CT$  take  $CN$  a third proportional to  $CT$  and  $CA$ .  $N$  will be the foot of the ordinate of the point of contact of the given tangent; i.e. if  $NP$  be drawn perpendicular to  $AA_1$  meeting  $TP$  in  $P$ ,  $P$  will be a point on the curve, and the problem therefore reduces to the preceding.

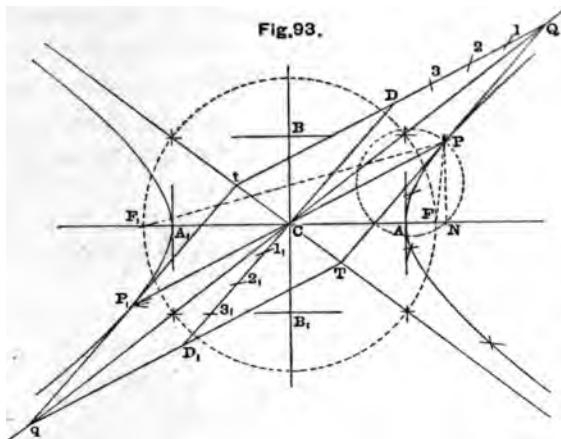
It may also be completed by Prob. 19, p. 23, determining two lines  $PF, PF_1$  making equal angles with  $PT$  and meeting  $AA_1$  in points equidistant from  $C$ ; since it has been already shewn (p. 157) that the tangent bisects the angle between the focal distances.

The proof follows from Prop. 4, p. 164, which of course applies to the principal axes.

PROBLEM 97. *To describe an hyperbola, a pair of conjugate diameters being given (Fig. 93).*

$PCP_1, DCD_1$  are the given conjugate diameters.

*First Method.* Complete the parallelogram  $QtqT$  formed by the tangents at their extremities; then the diagonals of this parallelo-



gram are the asymptotes (p. 163), and the axes therefore bisect the angles between them. Thus  $CA$  and  $CB$  are determined in direction.

From  $P$  draw two lines  $PF$ ,  $PF_1$  making equal angles with  $PT$ , the tangent at  $P$ , and meeting  $AA_1$  in points  $F$  and  $F_1$  equidistant from  $C$  (Prob. 19, p. 23). Then  $F$  and  $F_1$  are the foci, and the vertices can be determined by dropping perpendiculars on  $AA_1$  from the points in which the circle on  $FF_1$  as diameter intersects the asymptotes.

The curve can therefore be drawn by the general method.

*Second Method.* Points on the curve can also be determined without finding the foci thus:

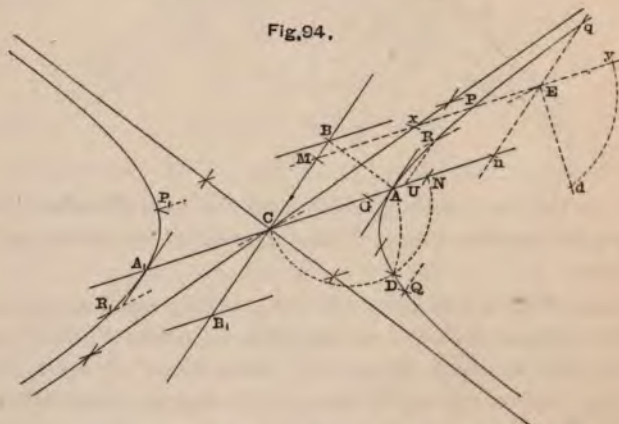
Complete the parallelogram  $QtqT$  as before.

Divide  $QD$  into any number of equal parts as at 1, 2, 3. Divide  $CD$  into the same number of equal parts as at  $1_1$ ,  $2_1$ ,  $3_1$ ; then  $P1$  and  $P_11_1$  will, when produced, intersect in a point on the curve, and *similarly with the other corresponding points.*

This method can also of course be applied to the principal axes; it cannot however be recommended, because a slight inaccuracy in the position of either line makes a considerable alteration in the position of the point on the curve, since in producing the lines the error is magnified, and the lines must often be produced to a considerable distance.

PROBLEM 98. *To describe an hyperbola, the centre  $C$ , the directions of a pair of conjugate diameters  $CA$ ,  $CB$ , and two points on the curve  $P$  and  $Q$  being given (Fig. 94).*

Draw  $PN$ ,  $Qn$  parallel to  $CB$  meeting  $CA$  in  $N$  and  $n$ . (Let  $PN$  be less than  $Qn$ ; then  $CN$  must be less than  $Cn$ .) Produce



$Qn$  to  $q$  and draw  $PMP_1$  parallel to  $CA$ , meeting  $CB$  in  $M$ . Make  $MP_1 = MP$  and  $nq = nQ$ . Then  $P_1$  and  $q$  are points on the curve. Let  $PP_1$  meet  $Qq$  in  $E$ . Through  $n$  draw  $nx$  parallel to  $QP_1$  meeting  $PP_1$  in  $x$ , and through  $n$  draw  $ny$  parallel to  $Pq$  meeting  $PP_1$  in  $y$ . Take  $Ed$  a mean proportional between  $Ex$  and  $Ey$ . On  $CN$  describe a semi-circle  $CDN$  and make  $ND = Ed$ .  $CD$  will be the length  $CA$  of the diameter parallel to  $PP_1$ . On  $CA$  make  $CG = Ed$ , and through  $A$  draw  $BA$  parallel to  $MG$ .  $CB$  will be the diameter conjugate to  $CA$ , and the problem reduces to the preceding.

*Proof.* The construction, as in the similar problem for the ellipse, depends on the property of the curve, that "the rectangles contained by the segments of any two chords which intersect each other are in the ratio of the squares on the parallel diameters,"

$$\text{i.e. } EP \cdot EP_1 : EQ \cdot Eq :: CA^2 : CB^2,$$

which may be thus proved :

Through  $E$  draw the diameter  $ERR_1$ , and draw the ordinate  $RU$  parallel to  $Qq$  or to  $CB$ ; then, by Prop. 5, p. 165,

$$RU^2 : CU^2 - CA^2 :: CB^2 : CA^2,$$

$$\therefore CB^2 + RU^2 : CB^2 :: CU^2 : CA^2,$$

and  $qn^2 : Cn^2 - CA^2 :: CB^2 : CA^2;$

$$\therefore CB^2 + qn^2 : CB^2 :: Cn^2 : CA^2,$$

so that  $CB^2 + RU^2 : CU^2 :: CB^2 + qn^2 : Cn^2.$

But  $RU^2 : CU^2 :: En^2 : Cn^2,$

$$\therefore CB^2 : CU^2 :: CB^2 + qn^2 - En^2 : Cn^2,$$

or  $CB^2 : CB^2 + qn^2 - En^2 :: CU^2 : Cn^2$   
 $:: CR^2 : CE^2;$

$$\therefore CB^2 : qn^2 - En^2 :: CR^2 : CE^2 - CR^2,$$

or  $CB^2 : EQ \cdot Eq :: CR^2 : ER \cdot ER_1.$

Similarly  $CA^2 : EP \cdot EP_1 :: CR^2 : ER \cdot ER_1,$

$$\therefore EP \cdot EP_1 : EQ \cdot Eq :: CA^2 : CB^2.$$

But by construction  $EQ : EP_1 :: En : Ex,$

and  $Eq : EP :: En : Ey,$

$$EQ \cdot Eq : EP \cdot EP_1 :: En^2 : Ex \cdot Ey;$$

but  $Ex \cdot Ey = Ed^2 = ND^2 = CN^2 - CA^2 = AN \cdot NA_1,$

$$\therefore En^2 \text{ or } PN^2 : AN \cdot NA_1 :: EQ \cdot Eq : EP \cdot EP_1,$$

which proves that  $AA_1$  is the diameter conjugate to  $PN$ .

Also by construction

$$CB : CA :: CM : CG :: PN : ND,$$

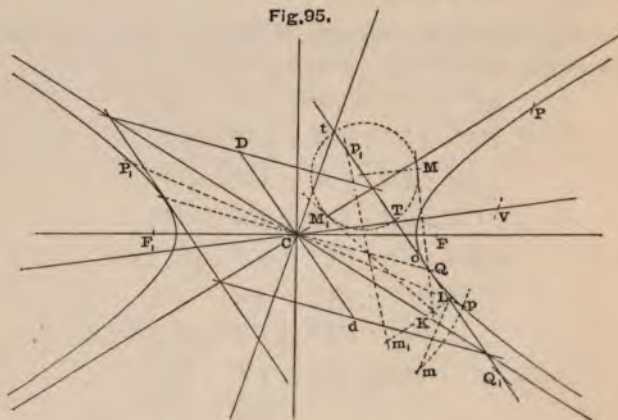
$$\therefore CB^2 : CA^2 :: PN^2 : AN \cdot NA_1,$$

or  $CB$  is the semi-diameter conjugate to  $CA$ .

PROBLEM 99. To describe an hyperbola, the centre  $C$ , the directions of a pair of conjugate diameters  $CT$ ,  $Ct$ , a tangent  $Tt$ , and a point  $P$  on the curve being given (Fig. 95).

[If a line be drawn parallel to  $Tt$  and at an equal distance from  $C$ , it will of course be a second tangent, and  $P$  must not

Fig. 95.



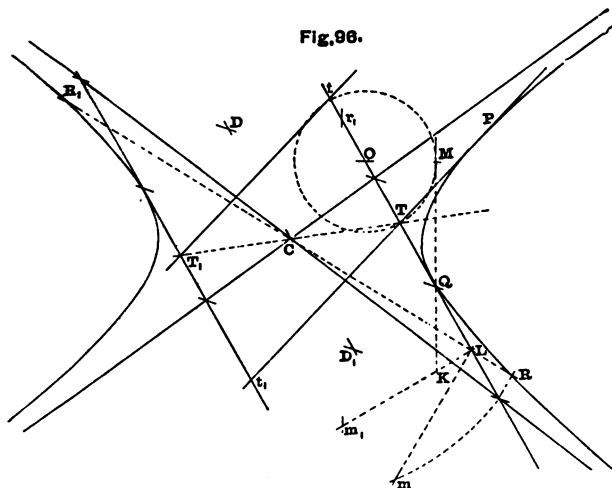
lie between these lines.] Draw  $PVp$  parallel to  $Ct$  cutting  $CT$  in  $V$ , and make  $Vp = VP$ .  $p$  will be a point of the curve. One of the two diameters  $CP$  or  $Cp$  (in the figure  $Cp$ ) will always intersect  $Tt$  in a point ( $L$ ) outside  $Tt$ ; draw such diameter and on it make  $CP_1 = Cp$ .  $P_1$  will be a point on the curve.

Take  $Lm$  a mean proportional between  $Lp$  and  $LP_1$ . On  $Tt$  as diameter describe a circle; through  $L$  draw  $LK$  perpendicular to  $Lt$  and on it take a point  $K$  such that  $Cp : Lm :: \frac{1}{2}Tt : LK$ , i.e. on  $LK$  make  $Lm_1 = Lm$ , and on  $Lt$  make  $Lo = \frac{1}{2}Tt$  and  $Lp_1 = Cp$ . Through  $o$  draw  $oK$  parallel to  $m_1p_1$ ,  $K$  will be the point required; then tangents  $KQM$ ,  $Q_1KM_1$  from  $K$  to the circle on  $Tt$  will intersect  $Tt$  in points ( $Q$ , and  $Q_1$ ) either of which may be taken for its point of contact with the curve. There are therefore two solutions.

Through  $C$  draw  $DCd$  parallel to  $Tt$ , make  $CD = Cd = QM$ , the tangent from  $Q$  to the circle.  $CQ$  and  $CD$  will be conjugate *semi-diameters*, and the problem reduces to Problem 97.

**PROBLEM 100.** To describe an hyperbola, the centre  $C$ , two tangents  $PT$ ,  $QT$  and a point on the curve ( $R$ ) being given (Fig. 96).

Through  $C$  draw  $TCT_1$ , and make  $CT_1 = CT$ . Draw  $T_1t$  parallel to  $PT$  meeting  $QT$  in  $t$ , and draw  $T_1t_1$  parallel to  $QT$  meeting  $PT$



in  $t_1$ .  $tT_1$  and  $t_1T_1$  will be tangents to the curve and  $TCT_1$ ,  $tCt_1$  will be the directions of a pair of conjugate diameters; and the problem therefore reduces to the preceding.

[The point  $R$  must lie outside the parallelogram  $TtT_1t_1$  and within one of the exterior angles, such as  $PTQ$ .]

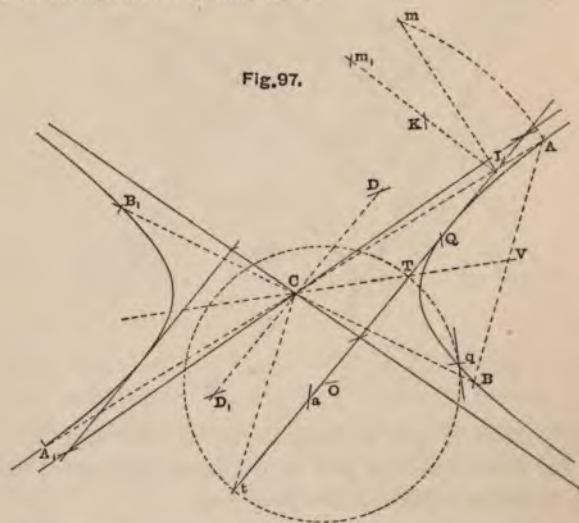
In the figure  $RC = CR_1$  and  $RC$  meets  $Tt$  in  $L$ ;  $Lm$  is a mean proportional between  $LR$  and  $LR_1$ ; and  $LK : Lm :: OT : CR$ , where  $OT = \frac{1}{2}Tt$ , and  $LK$  is perpendicular to  $Tt$ . Then a tangent  $KM$  from  $K$  to the circle on  $Tt$  as diameter cuts  $Tt$  in its point of contact ( $Q$ ) with the curve, and  $CD$  drawn through  $C$  parallel to  $QT$  and equal to  $QM$  will be the semi-diameter conjugate to  $CQ$ .

There are two solutions, as two tangents can be drawn from  $K$  to the circle on  $Tt$ .



PROBLEM 101. To describe an hyperbola, the centre  $C$ , two points ( $A$  and  $B$ ) of the curve and a tangent  $Tt$  being given (Fig. 97).

[A second tangent can at once be drawn parallel to  $Tt$  on the other side of  $C$  and at the same distance from it; the points  $A$  and  $B$  must not lie between these lines.] If the given points lie on opposite branches of the curve, as e.g.  $A$  and  $B_1$ , i.e. if they are on opposite sides of  $Tt$ , draw  $B_1CB$  and make  $CB = CB_1$ , then  $B$  will be on the same branch as  $A$ .



Draw  $AB$  and bisect it in  $V$ . Draw  $CV$  meeting the given tangent in  $T$ , and  $Ct$  parallel to  $AB$  meeting it in  $t$ . Then  $CT$ ,  $Ct$  are the directions of a pair of conjugate diameters, and the problem reduces to Prob. 99.

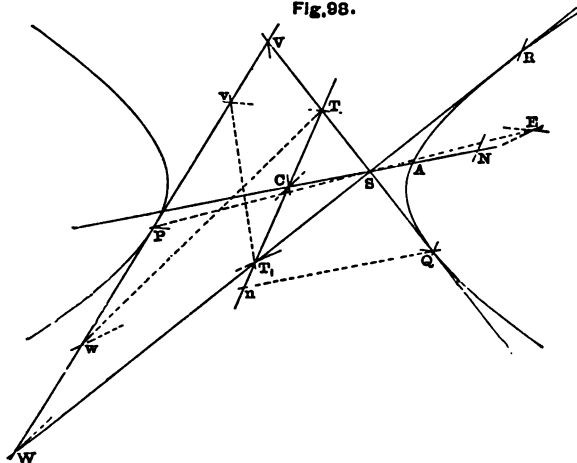
In the figure  $CA_1 = CA$  and  $ACA_1$  meets  $Tt$  in  $L$ ;  $Lm$  is a mean proportional between  $LA$  and  $LA_1$ ;  $LK : Lm :: OT : CA$ , where  $LK$  is perpendicular to  $Tt$  and  $OT = \frac{1}{2}Tt$ .

Then a tangent ( $Kq$ ) from  $K$  to the circle on  $Tt$  as diameter cuts  $Tt$  in  $Q$ , its point of contact with the curve.  $CD$  parallel to  $Tt$  and equal to  $QK$  is the semi-diameter conjugate to  $CQ$ .

**PROBLEM 102.** *To describe an hyperbola, the centre  $C$ , and three tangents ( $SV$ ,  $VW$ ,  $WS$ ) being given (Fig. 98).*

Through  $C$  draw  $TCT_1$  meeting  $SV$  in  $T$  and  $SW$  in  $T_1$ , so that  $TC = CT_1$  (Prob. 14, p. 19).  $CT$  will be conjugate to  $CS$ . Draw

Fig. 98.



$T_1v$  parallel to  $CV$  meeting  $VW$  in  $v$ ,  $Tw$  parallel to  $CW$  meeting  $VW$  in  $w$ , and draw  $vT$ ,  $wT_1$  meeting in  $E$ . Then  $ES$  will cut  $VW$  in  $P$ , its point of contact with the curve. Also  $PQ$  parallel to  $vT$  will cut  $VS$  in  $Q$ , its point of contact, and  $QR$  parallel to  $CT$  or  $PR$  parallel to  $T_1w$  will cut  $WS$  in  $R$ , its point of contact. The problem can be completed by several of those previously given or thus;

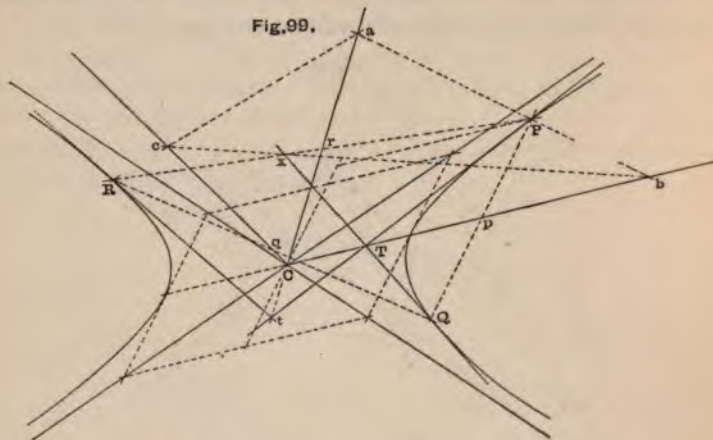
Draw  $QN$  parallel to  $CT$  meeting  $CS$  in  $N$ .  $QN$  is an ordinate of the diameter  $CS$ , and therefore  $CA$ , the length of the semi-diameter, is a mean proportional between  $CS$  and  $CN$  (Prop. 4, p. 164). Similarly, if  $Qn$  be drawn parallel to  $CS$  meeting  $CT$  in  $n$ ,  $CB$  must be taken as a mean proportional between  $Cn$  and  $CT$ .

**PROBLEM 103.** *To describe an hyperbola, the centre  $C$  and three points  $P$ ,  $Q$ ,  $R$  being given (Fig. 99).*

[Each of the points must lie either *between* both pairs of lines furnished by the remaining points and their corresponding points, or *outside* both these pairs of lines.]



Bisect  $PQ$  in  $p$ ,  $QR$  in  $q$ , and  $RP$  in  $r$ , and draw  $Cp$ ,  $Cq$  and  $Cr$ , producing each indefinitely.  $PQ$  is a double ordinate of the



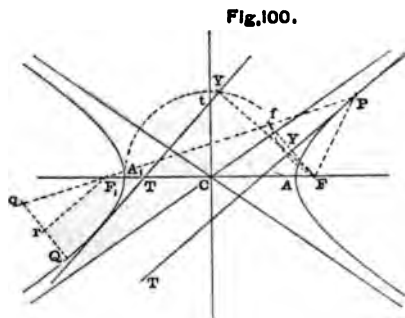
diameter  $Cp$ , and therefore the tangents at  $P$  and  $Q$  will intersect on  $Cp$ ; similarly the tangents at  $Q$  and  $R$  will intersect on  $Cq$ , and at  $R$  and  $P$  on  $Cr$ . If therefore a triangle be drawn (Prob. 15, p. 20), the sides of which pass through  $P$ ,  $Q$  and  $R$ , and the vertices of which lie on  $Cp$ ,  $Cq$  and  $Cr$  respectively, the sides of this triangle will be the tangents at  $P$ ,  $Q$  and  $R$ . Take *any* point  $a$  on  $Cr$ ; draw  $Pa$ ,  $Ra$  cutting  $Cp$ ,  $Cq$  in  $b$  and  $c$  respectively; join  $bc$  cutting  $PR$  in  $x$ , and draw  $xQ$  cutting  $Cb$  in  $T$ .  $QT$ ,  $PT$  will be the tangents at  $Q$  and  $P$  respectively; and if  $PT$  meet  $Cr$  in  $t$ ,  $Rt$  will be the tangent at  $R$ . The problem may be completed by preceding problems.

**PROBLEM 104.** To describe an hyperbola, the foci  $F$ ,  $F_1$  and a point  $P$  on the curve being given (Fig. 100).

It has been shewn already that the difference between the focal distances of any point on the curve is equal to the transverse axis (p. 153).

Let  $F_1P$  be greater than  $FP$ . On  $PF_1$  make  $Pf = PF$ . Draw  $FF_1$ , bisect it in  $C$  and make  $CA = CA_1 = \frac{1}{2}F_1f$ .  $AA_1$  will there-

fore be the vertices of the curve, and the problem reduces to Prob. 89.



**PROBLEM 105.** *To describe an hyperbola, the foci  $F, F_1$  and a tangent  $Tt$  being given (Fig. 100).*

[The tangent must lie between  $F$  and  $F_1$ .]

Bisect  $FF_1$  in  $C$ , the centre of the curve. From  $F$  or  $F_1$  drop a perpendicular (as  $FY$ ) on the tangent. On  $CF, CF_1$  make  $CA = CA_1 = CY$ .  $A$  and  $A_1$  will be the vertices of the curve, and the problem reduces to Prob. 89.

**PROBLEM 106.** *To describe an hyperbola, a focus  $F$ , a tangent  $PT$  with its point of contact  $P$ , and a second point  $Q$  on the curve, being given (Fig. 100).*

If  $F$  and  $Q$  are on the same side of  $PT$ , the solution has already been given in the corresponding problem for the ellipse (Prob. 76). Hence the case of  $F$  and  $Q$  lying on opposite sides of  $PT$  need alone be considered here.

From  $F$  draw  $FY$  perpendicular to  $PT$  meeting it in  $Y$ , on  $FY$  produced make  $Yf = YF$ ; then  $Pf$  will be a locus of the second focus. From  $f$  on  $fP$ , on either side of  $f$ , make  $fq = FQ$ . Bisect  $Qq$  in  $r$ , and draw  $rF_1$  perpendicular to  $Qq$  meeting  $Pf$  in  $F_1$ , which will be the second focus. Hence, both foci being known, the problem may be completed by Probs. 104 or 105.

Since  $q$  may be taken on either side of  $f$ , there are in general two solutions.

*Proof.* That  $fP$  is a locus of the second focus has been shewn in p. 157; that the second focus is at the intersection of  $fP$  and  $rF_1$  is evident thus:—it must be so situated that

$$F_1P \sim FP = FQ \sim F_1Q;$$

but

$$fQ = FQ \text{ and } F_1q = F_1Q,$$

$$\therefore FQ \sim F_1Q = fq \sim F_1q = fF_1,$$

and

$$FP = fP, \therefore F_1P \sim FP = F_1P \sim fP = fF_1,$$

i. e.  $F_1$  is the second focus.

If  $F$  and  $Q$  are on opposite sides of  $PT$ , two hyperbolas can in general be drawn.

If  $F$  and  $Q$  are on the same side of  $PT$ , and the distance of  $Q$  from  $F$  is greater than its distance from the line drawn through  $f$  perpendicular to  $Pf$ , two hyperbolas can in general be drawn.

If  $F$  and  $Q$  are on the same side of  $PT$ , but the distance of  $Q$  from  $F$  is less than its distance from the above perpendicular, one hyperbola only can in general be drawn, but an ellipse can also be drawn.

If  $F$  and  $Q$  are on the same side of  $PT$ , and the distance of  $Q$  from  $F$  is equal to its distance from the above perpendicular, a parabola can be drawn fulfilling the required conditions, but no hyperbola or ellipse, since the second focus removes to an infinite distance.

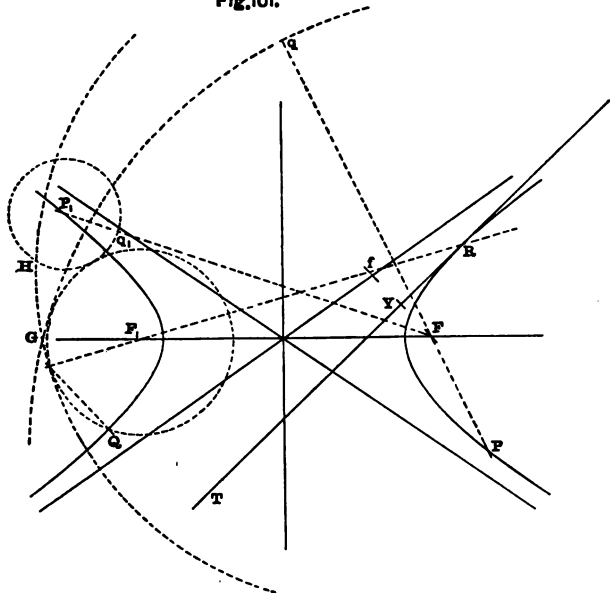
**PROBLEM 107.** *To describe an hyperbola, a focus  $F$ , a tangent  $RT$  and two points  $P$  and  $Q$  of the curve being given (Fig. 101).*

If  $F$ ,  $P$  and  $Q$  are all on the same side of  $RT$ , the solution has already been given in Prob. 77, the corresponding problem for the ellipse. Hence the cases of one or both of the points  $P$ ,  $Q$  lying on the opposite side of  $RT$  to  $F$  need be considered.

*Case 1.* Let  $F$  and  $P$  be on the same side of  $RT$ , and  $Q$  on the opposite side. Produce  $PF$  to  $q$ , make  $Fq = FQ$ , and with centre  $P$  and radius  $Pq$  describe a circle  $qG$ . From  $F$  draw  $FY$  perpendicular to  $RT$ , produce it to  $f$ , and make  $Yf = YF$ . With centre  $f$  and radius  $FQ$  describe a circle  $GH$ , and find the centre ( $F_1$ ) of a circle touching the circles  $qG$  and  $GH$  internally and

passing through  $Q$  (Prob. 32).  $F_1$  will be the second focus. The problem is always possible, since the circles must necessarily cut each other and the point  $Q$  be inside both.

**Fig. 101.**



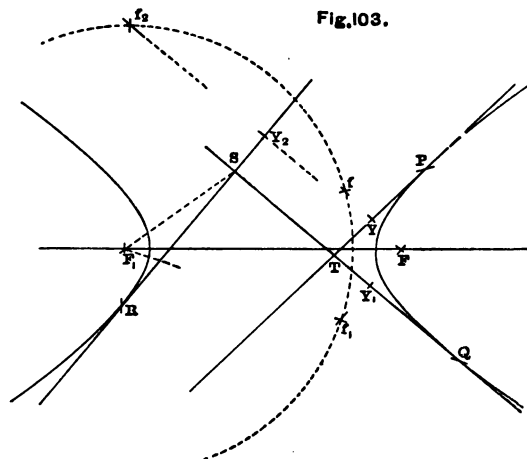
**Proof.** The second focus  $F_1$  must be equidistant from  $Q$  and from the circle  $qG$ , since  $F_1P - FP$  must be equal to  $FQ - FQ$ . But  $F_1P$  by construction  $= Pq - FQ$  and  $Pq = FP + Fq = FP + FQ$ .

Also  $fF_1$  must be equal to the transverse axis, p. 153, i.e. to  $FQ - F_1Q$  or to  $fG - F_1Q$ , i.e. the second focus must be equidistant from the point  $Q$  and from the circle  $GH$ .

**Case 2.** Let  $P$  and  $Q$  be on the opposite side of  $RT'$  to  $F$ , as  $P_1$  and  $Q$ . Let  $FP_1$  be greater than  $FQ$ . On  $FP_1$  make  $Fq_1 = FQ$ , and with centre  $P_1$  and radius  $P_1q_1$  describe the circle  $q_1H$ . Determine the point  $f$  as in Case 1, and with centre  $f$  and radius  $FQ$



**PROBLEM 109.** *To describe an hyperbola, a focus  $F$  and three tangents  $PT$ ,  $QT$  and  $RS$  being given (Fig. 103).*



[ $F$  must not lie within the triangle formed by the tangents.]

From  $F$  drop perpendiculars  $FYf$ ,  $FY_1f_1$ ,  $FY_2f_2$  on the given tangents, meeting them respectively in  $Y$ ,  $Y_1$  and  $Y_2$ , and make  $Yf = YF$ ,  $Y_1f_1 = Y_1F$ , and  $Y_2f_2 = Y_2F$ ; then, as  $f$ ,  $f_1$  and  $f_2$  must all be equidistant from the second focus (p. 153), and the problem therefore reduces to finding the centre ( $F_1$ ) of a circle passing through three given points (Prob. 20),  $F_1$  will be the second focus, and the transverse axis is of course known, since it is equal to  $F_1f$ .

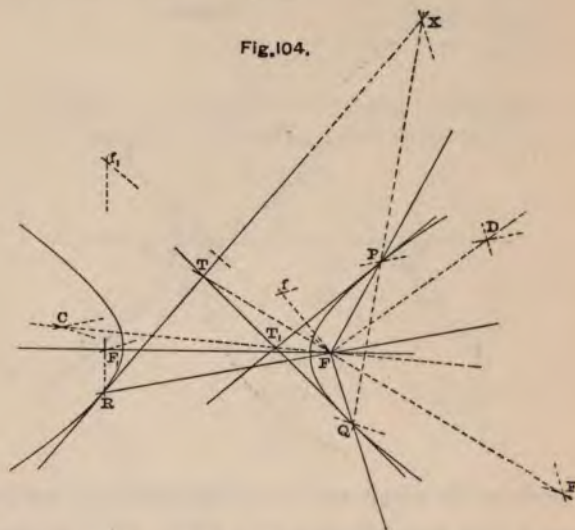
**PROBLEM 110.** *To describe an hyperbola, a focus  $F$  and three points  $P$ ,  $Q$ ,  $R$  on the curve being given (Fig. 104).*

[With two of the points as centres describe circles passing through  $F$ . The three given points cannot lie on the same branch of an hyperbola, unless

(1)  $F$  lies in one of the three angles  $PQR$ ,  $QRP$ , and  $RPQ$ ; and (2) the third point is more distant from  $F$  than it is from the common tangent to the above circles remote from  $F$ .



Whatever the relative position of the points and focus three hyperbolas can always be drawn.]



The points being as in the figure, the above conditions for the points lying on the same branch are not complied with; an ellipse and three hyperbolas can be drawn by the first solution of the corresponding problem in the preceding Chapter, Problem 80.

The second solution there given can be adapted to the present case thus: Let  $P$  and  $Q$  lie on one branch of the required hyperbola and  $R$  on the other.

Bisect the angle  $PFQ$  by the line  $FC$ , bisect the angle between  $PF$  and  $RF$  produced by  $FD$ , and the angle between  $QF$  and  $RF$  produced by  $FE$ .

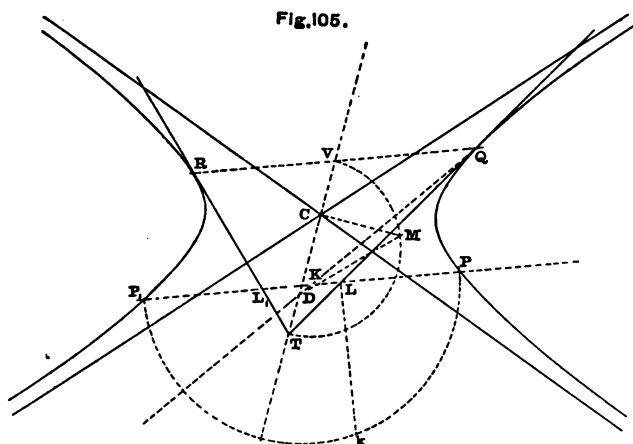
Determine the triangle whose sides pass through  $P$  terminating on  $FC$  and  $FD$ , through  $Q$  terminating on  $FE$ ,  $FC$ , and through  $R$  terminating on  $FD$  and  $FE$  (Problem 15). The sides of this triangle will be tangents to the required curve at  $P$ ,  $Q$  and  $R$  respectively.

*To determine the triangle.* Take any point  $C$  on  $FC$ , draw  $CP, CQ$  cutting  $CD, CE$  in  $D$  and  $E$ , and draw  $ED, QP$  intersecting in  $X$ .  $RX$  will be the tangent at  $R$ , and if it meets  $FE$  in  $T$ ,  $QT$  will be the tangent at  $Q$ ; similarly, if  $QT$  meets  $FC$  in  $T_1$ ,  $PT_1$  will be the tangent at  $P$ , passing also through the intersection of  $RT$  and  $DF$ .

The construction depends on the well-known property of the hyperbola, that the angles subtended at the focus by a pair of tangents are equal or supplementary according as the tangents touch the same or opposite branches of the curve.

**PROBLEM 111.** *To describe an hyperbola, two tangents  $TQ, TR$ , with their points of contact  $Q$  and  $R$ , and a point  $P$  on the curve being given (Fig. 105).*

[The point  $P$  must lie outside the parabola which can be described touching  $TQ, TR$  at  $Q$  and  $R$ .]



The construction is exactly similar to the corresponding problem for the ellipse. Prob. 81.

Bisect  $QR$  in  $V$  and through  $T$  draw  $TDV$ . Through  $P$  draw  $PLL_1$  parallel to  $QR$  meeting  $QT$  in  $L$  and  $RT$  in  $L_1$ .



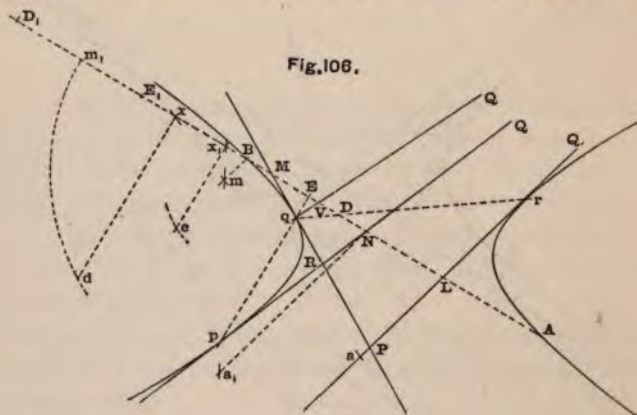
Find a mean proportional ( $Lk$ ) between  $PL$  and  $PL_1$ . (If  $P_1L_1$  be made equal to  $PL$ ,  $P_1$  will be a point on the curve.) On  $PP_1$  make  $PK = Lk$ , then  $QK$  will intersect  $TV$  in  $D$ , the extremity of the diameter  $TV$ .

On  $TV$  take a point  $C$  such that  $TC : CD :: CD : CV$ ; and if, as in the figure,  $Q$  and  $R$  are on opposite branches of the hyperbola,  $C$  must be taken between  $T$  and  $V$ ; i.e. on  $TV$  as diameter describe a semi-circle; draw  $DM$  making an angle of  $45^\circ$  with  $DV$  and meeting the semi-circle in  $M$ , and from  $M$  draw  $MC$  perpendicular to  $TV$ . Evidently  $MC$  is a mean proportional between  $CT$  and  $CV$ , and is equal to  $CD$ .  $C$  will be the centre of the hyperbola, and the asymptotes can easily be determined and the curve completed by preceding problems.

The proof is identical with that for the ellipse.

PROBLEM 112. To describe an hyperbola, two points  $A$  and  $B$  on the curve and three tangents  $PQ$ ,  $QR$ ,  $RP$  being given (Fig. 106).

[Either no one of the three tangents must pass between the



points or all three must do so, and the points must not lie within the triangle formed by the tangents.]

Draw a line through  $AB$  cutting the tangents through  $P$  in  $L$  and  $M$  and the remaining tangent in  $N$ .

Find  $X$  the centre and  $D, D_1$  the foci of the involution  $A, B$  and  $L, M$  (Prob. 13).  $D$  or  $D_1$  will be a point on the chord of contact of the tangents  $PL, PM$ .

[In the figure  $La$  on  $PQ = LA$ , and  $Bm$  on a parallel to  $PQ = BM$ ;  $am$  cuts  $AB$  in  $X$ , the required centre, and  $XD = XD_1 =$  a mean proportional between  $XM$  and  $XL$ .]

Similarly, find  $X_1$  the centre and  $E, E_1$  the foci of the involution  $A, B$  and  $M, N$  (Prob. 13), and  $E$  or  $E_1$  will be a point on the chord of contact of the tangents  $RM$  and  $RN$ .

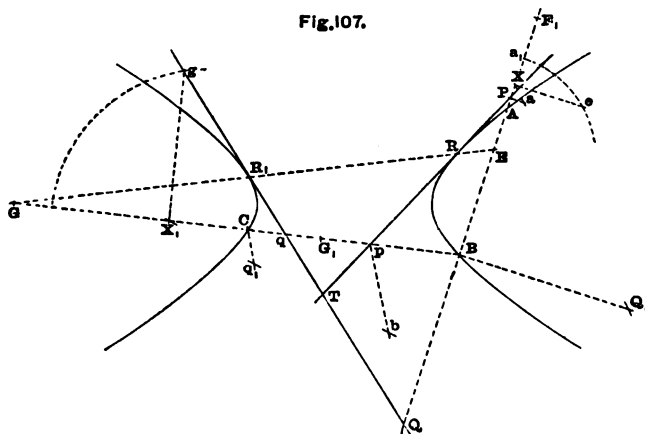
Find  $MV$  the harmonic mean between  $ME$  and  $MD$ ,  $M$  being the point of intersection of  $AB$  with the given tangents which has appeared in each of the above involutions, then  $QV$  ( $Q$  being the intersection of tangents through  $N$  and  $L$ ) will meet the tangent through  $M$  in its point of contact ( $q$ ) with the curve.

Therefore  $qDr$  will be the chord of contact of the tangents  $PQ, PR$ , and  $Epp$  the chord of contact of the tangents  $RP, RQ$ .

The proof is identical with that for the ellipse, p. 135.

PROBLEM 113. To describe an hyperbola, two tangents  $TP, TQ$  and three points  $A, B, C$  on the curve being given (Fig. 107).

[The points  $A, B, C$  being taken together in pairs, each pair



of points must be either both on the same side or both on opposite sides of both tangents. In the figure  $A$  and  $B$  are both on the same side, and  $B$  and  $C$  on opposite sides of both  $TP$  and  $TQ$ , as also  $C$  and  $A$ .]

Draw the line  $AB$  cutting the given tangents in  $P$  and  $Q$ . Find  $X$  the centre and  $E, E_1$  the foci of the involution  $A, B$  and  $P, Q$  (Problem 13).

[In the figure  $Pa = PA$  and  $BQ_1$  parallel to  $Pa = BQ$ .  $Q_1a$  cuts  $AB$  in  $X$ , the required centre.  $XE = XE_1 = a$  a mean proportional between  $XA$  and  $XB$ .]

$E$  or  $E_1$  will be a point on the chord of contact of the given tangents.

Again, draw  $BC$  cutting the given tangents in  $p$  and  $q$ , and find  $X_1$  the centre and  $G, G_1$  the foci of the involution  $B, C$  and  $p, q$ .  $G$  or  $G_1$  will be a second point on the chord of contact of the given tangents, the points of contact of which  $R, R_1$  are therefore determined, and the problem reduces to several preceding.

Since  $E$  and  $E_1$  can be joined to either  $G$  or  $G_1$  four chords of contact can in general be drawn, so that there are four solutions.

The construction depends on Prop. 7, p. 143.

PROBLEM 114. *To describe an hyperbola, five tangents  $AB, BC, CD, DE, EA$  being given (Fig. 108).*

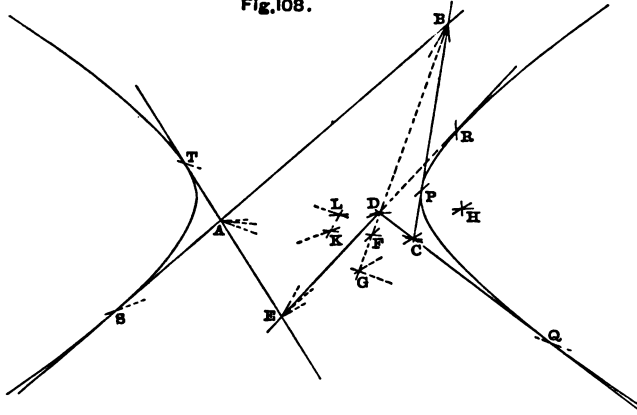
[The pentagon formed by the given tangents must contain a re-entering angle.]

Draw  $AC, BD$  intersecting in  $F$ ; and through the remaining angular point  $E$  of the pentagon draw  $EF$  meeting  $BC$  in  $P$ .  $P$  will be the point of contact of the given tangent  $BC$ . Similarly, if  $BD$  and  $CE$  intersect in  $G$ ,  $AG$  will intersect  $DC$  in  $Q$ , the point of contact of the given tangent  $CD$ ; and if  $CE$  and  $DA$  intersect in  $H$ ,  $BH$  will intersect  $ED$  in  $R$ , its point of contact.

The problem therefore reduces to Problem 111, or the points of contact  $S$  and  $T$  of the remaining tangents can easily be determined.

The construction depends (as in the corresponding problem for the ellipse, p. 136) on Brianchon's theorem.

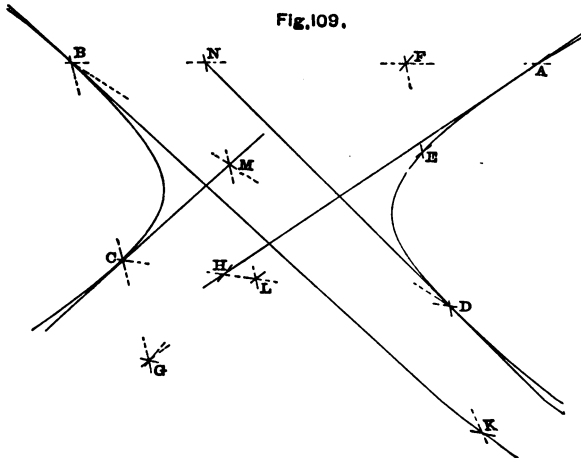
Fig. 108.



**PROBLEM 115.** *To describe an hyperbola, five points  $ABCDE$  being given (Fig. 109).*

Draw  $AB$ ,  $DE$  intersecting in  $F$ , and  $BC$ ,  $EA$  intersecting in  $G$ ; then, if  $FG$  meet  $CD$  in  $H$ ,  $H$  will be a point on the tangent at  $A$ , which can therefore be drawn.

Fig. 109.



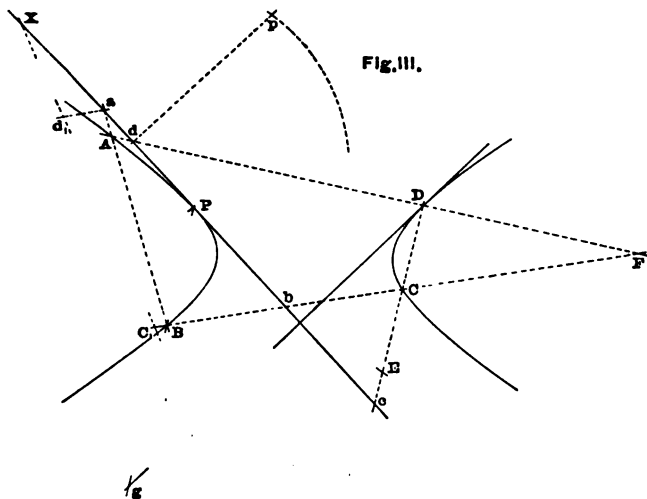


[In the figure  $Ba$  on  $BC = BA$ , and  $cd$  on a parallel to  $BC = cD_1$ ; then  $ad$  meets  $AB$  in  $X$ , the required centre of the involution, and  $XF = XF_1 =$  a mean proportional between  $XD_1$  and  $XB$ .

If  $FE$  meets  $DA$  in  $G$ , the points of contact of the given tangents may be determined by drawing  $GC$ ,  $DB$  intersecting in  $H$ , when  $FH$  will meet  $CD$  in  $P$ , the point of contact of  $CD$ ;  $GC$ ,  $DF$  intersecting in  $K$ , when  $BK$  will meet  $GD$  in  $Q$ , its point of contact;  $CF$  and  $GB$  intersecting in  $L$ , when  $DL$  will meet  $AB$  in  $R$ , its point of contact; the determination of the point of contact of  $CB$  is left as an exercise for the student.]

**PROBLEM 117.** To describe an hyperbola, four points  $A, B, C, D$  of the curve and a tangent  $ad$  being given (Fig. 111).

Let  $AB$  meet the given tangent in  $a$ , and  $BC, CD, DA$  meet it in  $b, c$ , and  $d$  respectively. Find  $X$  the centre and  $P, P_1$  the



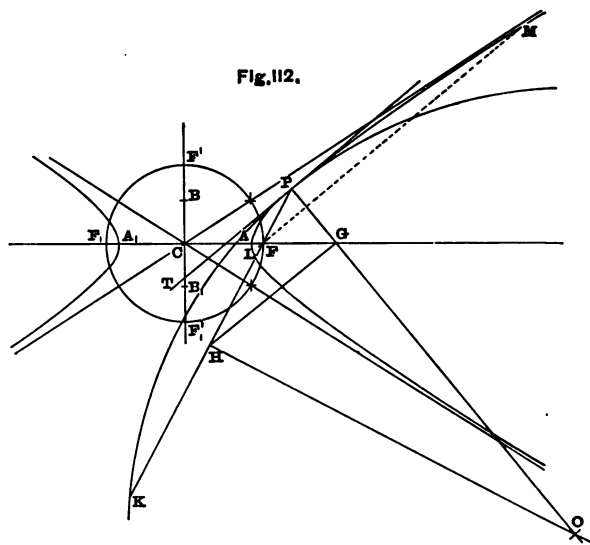
foci of the involution  $a, c$  and  $b, d$ ;  $P$  or  $P_1$  will be the point of contact of the given tangent, so that five points being known the problem reduces to Prob. 115.

[In the figure  $bc_1$  on  $bB = bc$ , and  $ad_1$  on a parallel to  $bB = ad$ ; then  $c_1d_1$  meets  $ad$  in  $X$ , the centre of the required involution, and  $XP$  is a mean proportional between  $Xd$  and  $Xb$ ;  $P_1$ , the other focus, of course lies outside the limits of the figure.

If  $AP$  and  $DC$  meet in  $E$ , and  $AD$ ,  $CB$  in  $F$ ,  $EF$  will meet  $PB$  in  $g$ , a point on the tangent at  $D$ .]

PROBLEM 118. *To find the centre and radius of curvature at any point  $P$  of a given hyperbola (Fig. 112).*

The construction is identical with that for the ellipse (Prob. 88).



Draw  $PG$  the normal meeting the axis in  $G$ ,  $GH$  perpendicular to  $PG$  meeting the focal chord  $PF$  in  $H$ , and  $HO$  perpendicular to  $PF$  meeting the normal in  $O$ , the required centre of curvature.

## CHAPTER VI.

### THE RECTANGULAR HYPERBOLA.

IF the axes of an hyperbola be equal, the angle between the asymptotes is a right angle, and the curve is called equilateral or rectangular.

If  $C$  is the centre,  $A$  a vertex, and  $F$  the corresponding focus, it follows that  $CF^2 = 2AC^2$ , for it has been shewn (p. 154) that  $CF^2 = CA^2 + CB^2$ , and in the rectangular hyperbola  $CB = CA$ .

Similarly  $FA^2 = 2AX^2$ , where  $X$  is the foot of the directrix, i.e. the *eccentricity* is always  $\sqrt{2} : 1$ .

Conjugate diameters are equal to one another and are equally inclined to either asymptote, for in any hyperbola

$$CP^2 - CD^2 = CA^2 - CB^2 \text{ and } \therefore CP = CD.$$

Also  $CPLD$  (fig. 113) is a rhombus and therefore  $CL$  bisects the angle  $PCD$ .

Diameters at right angles to one another are equal, for if  $CE$  be perpendicular to  $CP$  the angle  $BCE =$  the angle  $PCA =$  the angle  $BCD$ , and therefore by symmetry  $CE = CD$ .

**COROLLARY.** The rectangles contained by the segments of chords which intersect at right angles are equal since they are in the ratio of the squares of the parallel diameters (p. 169).

Given three points on an equilateral hyperbola, a fourth is also given, for if the curve pass through the three points  $A, B, C$  it will also pass through the *orthocentre* of the triangle  $ABC$ , i.e.



through the intersection of the perpendiculars from  $A, B, C$  on the opposite sides.

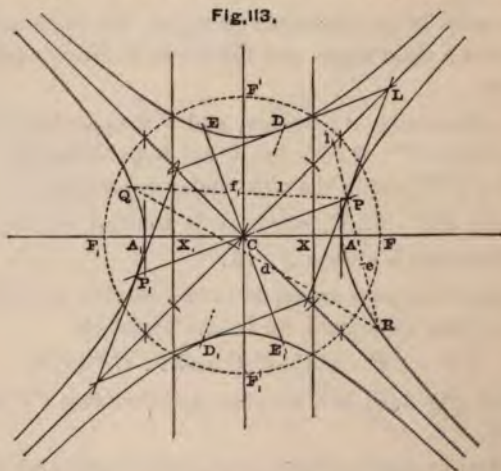
This follows at once from the above corollary, for if  $ABC$  be a triangle, and  $O$  the orthocentre, and if  $CO$  meets  $AB$  in  $D$ , the triangles  $DOA$  and  $DBC$  are similar, and

$$DO : DA :: DB : DC ;$$

$$\therefore DO \cdot DC = DA \cdot DB,$$

so that  $O$  must be a point on the curve.

If  $P, Q, R$  are three points on the curve, the centre must lie on the circle passing through the middle points of the sides of the triangle  $PQR$ . For (fig. 113) let an asymptote meet the sides



$PQ, PR$  in  $l$  and  $l_1$ , and let  $d, e, f$  be the middle points of  $QR, RP$  and  $PQ$  respectively. Let  $C$  be the centre of the hyperbola. Then  $Cf$  is conjugate to  $PQ$  and  $Ce$  to  $PR$ ,

$$\begin{aligned} \therefore \text{the angle } fCe &= fCl + l_1Ce \\ &= el_1C + Clf \\ &= Pl_1l + Pll_1 \\ &= fPe \\ &= fde, \end{aligned}$$

since  $fdeP$  is a parallelogram, i.e. the circle through  $fde$  passes also through  $C$  (Euclid III. 21).

Four points are therefore in general sufficient to determine a rectangular hyperbola, for the orthocentre of the triangle formed by any three is necessarily a fifth point on the curve, which can then be completed by the general method of Prob. 115, p. 185; or the centre can at once be determined as one of the points of intersection of the two circles which can be described through the centre points of the sides of the triangles formed by taking any three of the four given points in succession.

Similarly a rectangular hyperbola can generally be determined from four conditions, and the curve cannot in general be described to satisfy a greater number.

If  $QV$  be an ordinate of a diameter  $PCP_1$ ,  $QV^2 = PV \cdot VP_1$ .

For in any hyperbola (Prop. 5, p. 165)

$$QV^2 : PV \cdot VP_1 :: CD^2 : CP^2;$$

but in the rectangular hyperbola  $CD = CP$ ,

$$\therefore QV^2 = PV \cdot VP_1.$$

Hints for the solution of particular cases are given in the following examples, but as they are usually simple it has not been considered necessary to illustrate them by figures.

Given the following data, construct rectangular hyperbolas fulfilling them.

a. An asymptote and focus.

[A line through the focus making  $45^\circ$  with the asymptote meets it in the centre.]

b. An asymptote  $LC$ , a tangent  $PL$  and its point of contact  $P$ .

[Let the given tangent  $PL$  meet the asymptote in  $L$ ; on it make  $PL_1 = PL$ , and draw  $L_1C$  perpendicular to the asymptote meeting it in  $C$ , the centre of the curve.]

c. The centre  $C$ , a tangent  $PT$  and its point of contact  $P$ .

[From  $C$  draw  $CY$  perpendicular to  $PT$  meeting it in  $Y$ .

The transverse axis bisects the angle  $PCY$ , and its semi-length  $CA$  is a mean proportional between  $CP$  and  $CY$ .]

d. The centre  $C$  and two points  $P, Q$  on the curve.

[Produce  $PC$  to  $p$  and make  $Cp = CP$ , so that  $Pp$  is a diameter. Describe a circle through the three points  $P, Q, p$ . The tangent to this circle at  $Q$  is parallel to the tangent to the curve at  $P$ , which is therefore known.]

e. The centre  $C$ , a tangent  $PT$  and a point  $Q$  of the curve.

[From  $Q$  drop a perpendicular  $QN$  on the given tangent, meeting it in  $N$ ; bisect  $QN$  in  $n$ , and draw  $nt$  parallel to  $PT$ . A circle passing through  $C$  and  $N$  and touching  $nt$  will meet  $PT$  again in its point of contact.]

f. The centre  $C$  and two tangents  $PT, QT$ .

[Produce  $TC$  to  $T_1$  and make  $CT_1 = CT$ ; through  $T_1$  draw  $T_1t$  parallel to  $QT$  and meeting  $PT$  in  $t$ .  $CT$  and  $Ct$  will be the directions of a pair of conjugate diameters, which determine the asymptotes.]

g. A focus  $F$  and two points  $P, Q$ .

[With centre  $P$  describe a circle, the radius of which

$$: FP :: 1 : \sqrt{2},$$

and with centre  $Q$  describe a circle, the radius of which

$$: FQ :: 1 : \sqrt{2}.$$

The directrix will be a common tangent to these two circles.]

h. A focus  $F$ , a tangent  $PT$  and its point of contact  $P$ .

[With centre  $P$  describe a circle, the radius of which

$$: FP :: 1 : \sqrt{2};$$

draw  $FT$  perpendicular to  $PT$  meeting it in  $T$ . A tangent from  $T$  to the circle will be the directrix.]

i. A focus  $F$  and two tangents  $PT, QT$ .

[From  $F$  draw  $GY$  perpendicular to  $PT$  meeting it in  $Y$ ; produce  $FY$  to  $f$ , and make  $Yf = YF$ . Draw the circle which is the

locus of the vertex of the triangle on base  $Ff$ , and with the sides terminating in  $F$  and  $f$  respectively in the ratio of  $\sqrt{2} : 1$ , (Prob. 17, p. 21). This circle is a locus of the second focus. Similarly the tangent  $QT$  will furnish a second locus, so that the second focus must be at one of the points of intersection of the two circular loci.]

*k.* A focus  $F$ , a tangent  $QT$ , and a point  $P$ .

[From  $F$  draw  $FY$  perpendicular to  $QT$ , produce it to  $f$ , and make  $Yf = YF$ ; a circular locus of the second focus can be determined from  $Ff$  as in the last example. With  $P$  and  $f$  as foci and with distance between the vertices  $= FP$ , describe an hyperbola, which will be a second locus of the second focus, which is therefore at one of the intersections of the hyperbola and circle.]

*l.* Two tangents  $PT, QT$  and their points of contact  $P$  and  $Q$ .

[Bisect  $PQ$  in  $V$ ; then  $VT$  is a locus of the centre. On  $PQ$  describe a segment of a circle containing an angle equal to the supplement of the angle  $PTQ$ ; the segment must be on the same side of  $PQ$  as  $T$ , and is then a second locus of the centre, which is therefore known.]

*m.* Given three points and a tangent.

[From the three points a fourth can be determined (p. 189), and the curve can be drawn by the general method. Prob. 117, p. 187.]

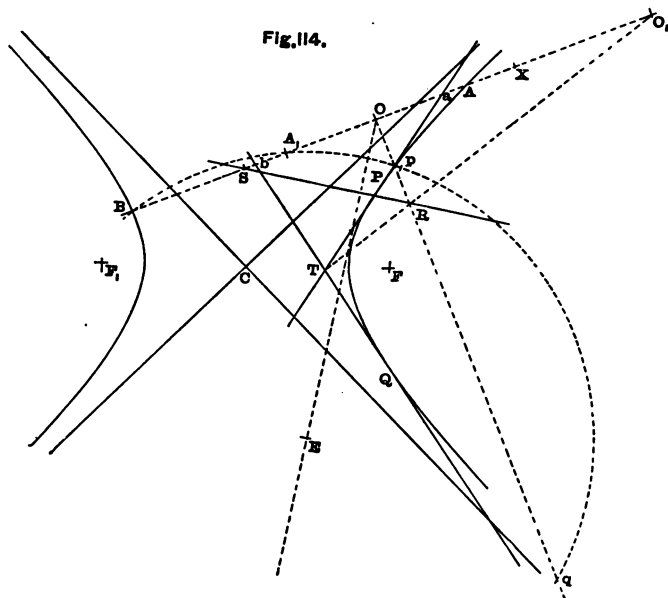
*n.* Given four tangents,  $AB, BC, CD, DA$ .

[Draw the circle to which the triangle formed by some three of the four tangents is self conjugate (Ex. 18, p. 55); it is a locus of the centre. A second locus is the straight line joining the points of bisection of the diagonals of the quadrilateral formed by the tangents. The centre is therefore known.]

*o.* Given two points  $A, B$  and two tangents  $PT, QT$ . (Fig. 114.)

[Let  $AB$  meet  $PT$  in  $a$  and  $QT$  in  $b$ . Find  $X$  the centre, and  $O, O_1$  the foci of the involution  $A, B$  and  $a, b$ . The chord of

contact of  $PT$  and  $QT$  will pass through  $O$  or  $O_1$ , and if it passes through  $O$ ,  $TO_1$  will be the polar of  $O$  and conversely. Through

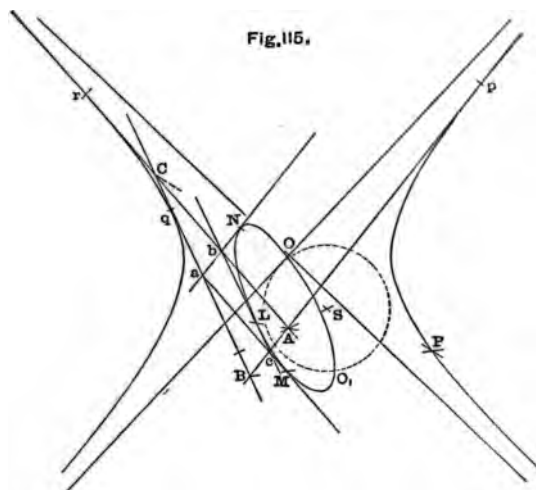


$O$  draw  $Opq$  perpendicular to  $AB$ . On  $AB$  make  $OA_1 = OA$ , and find  $OS$  an harmonic mean between  $OA_1$  and  $OB$ , let  $Oq$  and  $TO_1$  meet in  $R$  and join  $SR$ . A circle described with its centre on  $OE$  perpendicular to  $SR$  and to pass through  $A_1$  and  $B$  will cut  $Opq$  in points  $p, q$  of the required curve.]

*p.* Given three tangents  $AB, BC, CA$  and a point  $P$ . (Fig. 115.)

[The circle (centre  $S$ , radius  $SO$ ) to which the given triangle  $ABC$  is self-conjugate (Ex. 18, p. 55) is one locus of the centre. Bisect  $AB, BC, CA$  in  $c, a, b$  respectively, and draw  $PA$  cutting  $bc$  in  $L$ ,  $PB$  cutting  $ca$  in  $M$ , and  $PC$  cutting  $ab$  in  $N$ . The conic described to touch the sides of the triangle  $abc$  in the points  $LMN$  (Probs. 81 or 111) is a second locus of the centre. In the

figure, the required conic is an ellipse and the circular locus cuts



it in the points  $O$  and  $O_1$ , either of which may be taken as the centre. The points of contact of the given tangents are  $p$ ,  $q$ ,  $r$ .]

# EXAMPLES ON CHAPTERS V. AND VI.

1. Draw an hyperbola, the centre ( $C$ ), one asymptote ( $CL$ ) and a directrix  $LL_1$  being given.

[Draw the axis  $CF$  perpendicular to  $LL_1$ , meeting it in  $X$ . The vertex  $A$  is at a distance  $CL$  from  $C$ .]

2. Draw an hyperbola, the asymptotes  $CL$ ,  $CL_1$  and the distance  $CF$  of a focus  $F$  being given.

3. Given the base  $AB$  of a triangle and point of contact,  $F$ , with base of the inscribed circle; shew that the locus of vertex of triangle is an hyperbola with foci  $A$  and  $B$  and vertex  $F$ .

4. Shew that the tangent at any point  $P$  of an hyperbola bisects any straight line perpendicular to the axis  $AA_1$  and terminated by  $AP$ ,  $A_1P$ .

5. Draw the locus of the foci of parabolas passing through two fixed points  $P$  and  $P_1$  and having their axes parallel to a fixed line  $AB$ .

[The hyperbola described with  $P$  and  $P_1$  as foci and with length of transverse axis  $= PM$ ; the side parallel to  $AB$  of a right-angled triangle on  $PP_1$  as hypotenuse.]

6. Given the centre  $C$ , vertex  $A$ , and a tangent  $PT$  meeting  $CA$  in  $T$ , describe the hyperbola.

[The foot  $N$  of the ordinate of the point of contact ( $P$ ) may be determined from  $CT : CA :: CA : CN$ .  $P$  is then known. The asymptotes cut off equal distances on  $PT$  on each side of  $P$  and make equal angles with  $CA$  (Prob. 19).]

7. Given the centre  $C$ , the axis  $CT$ , a tangent  $PT$  and its point of contact  $P$ , draw the hyperbola. (See last example.)

8. Determine the locus of the intersection of the bisectors of the sides of the triangle formed by the asymptotes and any tangent to a hyperbola.

[A similar and similarly placed hyperbola with axes reduced in ratio 2 : 3.]

9. Given a focus  $F$ , tangent  $PT$  and point  $Q$  on a hyperbola, draw the locus of the second focus.

[From  $F$  draw  $FY$  perpendicular to  $PT$  meeting it in  $Y$ : produce  $FY$  to  $f$  and make  $Yf = FY$ . The required locus is the hyperbola with foci  $f$  and  $Q$ , and transverse axis  $= FQ$ .]

10. Given a line  $QT$  and two points  $P$  and  $F$ . From  $F$  draw a perpendicular  $FY$  to  $QT$  meeting it in  $Y$ , and produce  $FY$  to  $f$ , making  $Yf = YF$ . With  $P$  and  $f$  as foci and with  $PF$  as the distance between the vertices describe an hyperbola. With  $F$  and any point on this hyperbola as foci describe an ellipse to pass through  $P$ , and shew that it will touch  $QT$ :

i.e. Given a focus, tangent and point of a conic, the locus of the second focus is an hyperbola.



11. Given two tangents  $PT$ ,  $QT$  to a rectangular hyperbola and their points of contact  $P$ ,  $Q$ . Shew that if  $QR$  be drawn perpendicular to  $PT$ , and  $PR$  to  $QP$ ,  $R$  will be a point on the curve.

12. In a given ellipse determine the pair of equal conjugate diameters.

[They coincide with asymptotes of hyperbola having the same axes.]

13. Draw the loci of the points of trisection of a series of circular arcs described on the straight line  $AB$ .

[Branches of two hyperbolas having their centres at the internal points of trisection of  $AB$  and asymptotes inclined  $60^\circ$  to axis.]

14. Given the asymptotes and a point on a directrix, draw the hyperbola.

15. From a given point  $P$  in an hyperbola draw a straight line, such that the segment intercepted between the other intersection with the hyperbola and a given asymptote shall be equal to a given line.

[With  $P$  as centre and the length of the given line as radius describe a circle cutting the other asymptote. Either point of intersection joined to  $P$  gives the line required.]

16. Given a focus  $F$ , and tangent  $PY$  to an hyperbola and the length  $2a$  of the transverse axis, shew that the locus of the second focus is a circle.

[From  $F$  draw  $FY$  perpendicular to  $PY$  meeting it in  $Y$ ; produce  $FY$  to  $f$  and make  $Yf = FY$ .  $f$  is the centre and  $2a$  the radius of the required circle.]

17. Shew that any point on the circle through the middle points of the sides of a triangle  $ABC$  may be taken as the centre of an equilateral hyperbola passing through  $A$ ,  $B$  and  $C$ .



18. If four tangents to an equilateral hyperbola be given, shew that either of the *limiting points* (p. 46) of the system of circles described on the diagonals of the quadrilateral as diameters may be taken as the centre of the hyperbola.

19. Given a focus  $F$ , a tangent  $PT$ , its point of contact  $P$ , and the eccentricity, draw the conic.

[From  $F$  draw  $FT$  perpendicular to  $FP$  and meeting  $PT$  in  $T$  which will be a point on the directrix. With  $P$  as centre and with radius  $r$  such that  $\frac{FP}{r} =$  the given eccentricity, describe a circle. Tangents from  $T$  to this circle will be positions of the directrix. Two solutions are generally possible.]

20. Draw normals to an ellipse, from a given point  $P$ .

[The normals pass through the intersections of the ellipse with the rectangular hyperbola passing through  $P$  and the centre of the ellipse, and having its asymptotes parallel to the major and minor axes at distances respectively

$$-\frac{b^2\beta}{a^2-b^2}, \text{ and } \frac{a^2\alpha}{a^2-b^2},$$

where  $a$  and  $b$  are the semi-axes and  $\alpha, \beta$  the co-ordinates of  $P$ .]

21. Draw normals to an ellipse from a point on the minor axis.

[They will pass through the intersections of the ellipse with the circle described through the foci and point.]

## CHAPTER VII.

### RECIPROCAL POLARS AND THE PRINCIPLE OF DUALITY.

IN page 31, it has been shewn how to find the pole of a given line and the polar of a given point with regard to a given circle, and the principal properties of poles and polars have been explained.

IN pages 140 et seq. an extension has been made to the case of an ellipse, and the properties there noticed are applicable to all conic sections.

The pole of a line with regard to any conic being a point and the polar of a point a line, it follows that any system of points and lines can be transformed into a system of lines and points.

This process is called *reciprocation*, and it is clear that any theorem relating to the original system will have its analogue in the system formed by reciprocation.

DEF. Being given a fixed conic section ( $\Sigma$ ) and any curve ( $S$ ), we can generate another curve ( $s$ ) as follows; draw any tangent to  $S$ , and take its pole with regard to  $\Sigma$ ; the locus of this pole will be a curve  $s$ , which is called the *reciprocal polar* of  $S$  with regard to  $\Sigma$ . The conic  $\Sigma$  with regard to which the pole is taken is called the *auxiliary conic*.

A point (of the reciprocal polar curve  $s$ ) is said to *correspond* to a line (of the reciprocated figure  $S$ ) when we mean that the point is the pole of the line with regard to the auxiliary conic  $\Sigma$ ; and since it appears from the definition that every point of  $s$  is

the pole with regard to  $\Sigma$  of some tangent to  $S$ , this is briefly expressed by saying that every point of  $s$  corresponds to some tangent of  $S$ .

**THEOREM.** *The point of intersection of two tangents to  $S$  will correspond to the line joining the corresponding points of  $s$ .*

This follows from the property of the conic  $\Sigma$ , that the point of intersection of any two lines is the pole of the line joining the poles of these two lines. (p. 141.)

Now if the two tangents to  $S$  be indefinitely near, then the two corresponding points of  $s$  will also be indefinitely near, and the line joining them will therefore be a tangent to  $s$ ; and since any tangent to  $S$  intersects the consecutive tangent at its point of contact, the above theorem becomes: *If any tangent to  $S$  correspond to a point on  $s$ , the point of contact of that tangent to  $S$  will correspond to the tangent through the point on  $s$ .*

Hence we see that the relation between the curves is *reciprocal*, that is to say, that the curve  $S$  might be generated from  $s$  (through the auxiliary conic) in precisely the same manner that  $s$  was generated from  $S$ . Hence the name "reciprocal polars\*."

Being given then any theorem of *position* concerning any curve  $S$  (i.e. one not involving the magnitudes of lines or angles), another can be deduced concerning the curve  $s$ . For example, if we know that a number of points connected with the figure  $S$  lie on a right line, we know also that the corresponding lines connected with the figure  $s$  meet in a point (the pole of the line with regard to  $\Sigma$ ), and *vice versa*.

From any one such theorem another can be derived by suitably interchanging the words "point" and "line," "inscribed" and "circumscribed," "locus" and "envelope," &c., understanding by the term envelope "the curve to which a series of lines drawn according to any given rule are tangents."

The *evolute* of a curve, e.g. is the *envelope* of normals to the curve.

\* Salmon's *Conic Sections*, chap. xv.

Although the auxiliary conic  $\Sigma$  has hitherto been spoken of as any conic whatever, it is most common to make this conic a circle, considerable simplification being thereby introduced, and generally unless the contrary is specially mentioned, reciprocal polars may be understood to be polars *with regard to a circle*. It has been shewn, p. 31, that the polar of any point with regard to a circle is a line perpendicular to the line joining the point to the centre, and conversely that the pole of any given line with regard to a circle is on the line through the centre perpendicular to the given line; in either case the product of the distances of the pole and polar from the centre being equal to the square of the radius, so that the polar of a given point or the pole of a given line with regard to a given circle may always be found by merely drawing tangents to that circle. The centre of the auxiliary circle is frequently called the *origin*.

The advantage of using a circle for the auxiliary conic chiefly arises from the two following theorems, which enable us to transform by this method, not only theorems of position, but also theorems involving the magnitude of lines and angles.

**THEOREM.** *The distance of any point  $P$  from the origin  $O$  is the reciprocal of the distance  $Ot$  from the origin of the corresponding line  $pt$ ;*

$$\text{i. e. } OP \cdot Ot = r^2,$$

where  $Ot$  is perpendicular to  $pt$  and  $r$  is the radius of the auxiliary circle.

**THEOREM.** *The angle  $TQT_1$  between any two lines  $TQ$ ,  $T_1Q$  is equal to the angle subtended at the origin by the corresponding points  $p$ ,  $p_1$ ; for  $Op$  is perpendicular to  $TQ$  and  $Op_1$  to  $T_1Q$ .*

**PROBLEM 119.** To find the polar reciprocal of one circle, centre  $C$ , radius  $CA$ , with regard to another, centre  $O$ , radius  $OM$ , i. e. to find the locus of the pole  $p$  with regard to the circle  $(O)$  of any tangent  $PT$  to the circle  $C$ . (Fig. 116.)

Find  $MM_1$ , the polar with respect to the auxiliary circle (centre  $O$ ) of  $C$ , the centre of the circle to be reciprocated; i. e. if



Therefore the point  $p$  moves so that its distance ( $Op$ ) from a fixed point  $O$  is in a constant ratio to its distance ( $pN$ ) from a fixed right line  $MM_1$ ; i.e. the locus of  $p$  is a conic section of which  $O$  is a focus,  $MM_1$  the corresponding directrix, and the eccentricity of which is  $\frac{OC}{CP}$ . The eccentricity is evidently greater, less than, or equal to unity according as  $O$  is outside, inside, or on the circumference of the reciprocated circle.

Hence, *the polar reciprocal of a circle is a conic section, of which the origin is the focus, the line corresponding to the centre is the directrix, and which is an hyperbola, an ellipse, or a parabola, according as the origin is outside, inside, or on the circle.*

The tangents at  $A$  and  $A_1$ , the extremities of the diameter through  $O$ , correspond to the vertices at  $a$  and  $a_1$  of the reciprocal polar. [In the figure  $At$  touches the auxiliary circle and  $at$  is perpendicular to  $OC$ ].

The extremities of the latus rectum  $LL_1$  correspond to the tangents parallel to  $OC$ . Therefore  $OL \cdot CP = r^2$ , where  $r$  is the radius of the auxiliary circle.

The centre of the reciprocal conic is the pole with respect to ( $O$ ) of the polar of  $O$  with respect to ( $C$ ), i.e. if  $O$  is outside ( $C$ ) it is the pole of the chord of contact of tangents from  $O$  to the circle ( $C$ ), and in that case the asymptotes are perpendicular to these tangents. Of course if  $O$  is inside ( $C$ ) real tangents from it to ( $C$ ) cannot be drawn, and consequently the ellipse has no real asymptotes.

Conversely of course the reciprocal of a conic section with regard to a circle which has one of the foci for its centre is a circle, with its centre at the pole of the corresponding directrix and of radius ( $R$ ) such that the ratio,  $R$  : distance between its centre and the focus, is the eccentricity of the conic.

The above important property enables us to deduce from any property of a circle, a corresponding property of a conic; and since the proof of the existence of the relation in the circle will usually be much simpler than a direct proof of the corresponding



relation in the conic, the method is frequently of great value. It will soon be found that the operation of forming the reciprocal theorem will reduce itself to a mere mechanical process of interchanging the words "point" and "line," "inscribed" and "circumscribed," "locus" and "envelope," &c., as has been already noticed; but the method also furnishes admirable examples and tests of draughtsmanship, and the actual construction of reciprocal figures should I think be much more largely practised than it is.

Of course a little care is required in taking the original circles so that the resulting conic may be of convenient proportions, but a very little practice will enable this to be done and there is no real difficulty in the construction itself.

A convenient ratio for the eccentricity of an ellipse is one not very different from 3 : 4, which may therefore be taken as a guide for the ratio of the radius of the circle to be reciprocated to the distance of the origin from its centre; and the auxiliary circle should then be taken of such radius as to bring the length between the poles of the tangents at the extremities of the diameter through the origin, i.e. the length of the major axis, a convenient one. The approximate position of these poles relatively to any assumed radius is easily seen. The *size* of the reciprocal conic depends entirely on the radius of the auxiliary circle.

As an example, fig. 117 gives the figure illustrating the following reciprocal theorems:

## THEOREM.

If a chord of a circle subtend a constant angle at a fixed point on the curve, the chord always touches a circle.

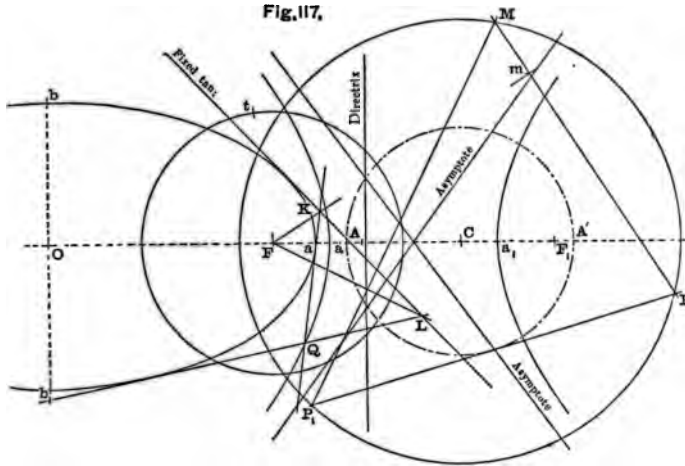
## RECIPROCAL.

If two tangents to a conic move so that the intercepted portion of a fixed tangent subtends a constant angle at the focus, the locus of the intersection of the moving tangents is a conic having the same focus and directrix.

$C$  is the centre of the circle,  $M$  the fixed point on it, and  $PP_1$  the chord which moves so that the angle  $PMP_1$  is constant, and *which therefore* always touches a circle described with centre  $C$ .

$F$  is the centre of the auxiliary circle, and since it is taken inside the circle ( $C$ ) the reciprocal polar of this circle will be an ellipse.

Find  $K$  the point corresponding to the line  $MP$ , i.e. the pole



of  $MP$  with respect to the auxiliary circle, (in other words draw  $Fm$  perpendicular to  $MP$  meeting it in  $m$ , draw  $mt$  a tangent to the auxiliary circle touching it in  $t$  and draw  $tK$  perpendicular to  $Fm$  meeting it in  $K$ ); find  $L$  the point corresponding to  $MP_1$ , i.e. the pole of  $MP_1$  with respect to the auxiliary circle, (in other words, since  $MP_1$  cuts the auxiliary circle, draw a tangent at one of the points of intersection meeting  $FL$  drawn perpendicular to  $MP_1$  in  $L$ ); then the line  $KL$  corresponds to the point  $M$ , and will therefore be a fixed tangent to the reciprocal conic.

Find  $Q$  the point corresponding to the line  $PP_1$ , then the line  $KQ$  corresponds to the moving point  $P$ , and the line  $LQ$  to the moving point  $P_1$ , and these lines are therefore moving tangents to the reciprocal conic, intercepting on the fixed tangent a length  $KL$  which subtends a constant angle at  $F$ , for since  $FK$  is perpendicular to  $MP$  and  $FL$  to  $MP_1$ , the angle  $KFL$  is equal to the angle  $PMP_1$ , which by supposition is constant.



Lastly, since  $Q$  corresponds to  $PP_1$  which is a tangent to a circle centre  $C$ , the locus of  $Q$  must be the polar reciprocal of this circle, and is therefore a conic with focus  $F$  and the polar of  $C$  for directrix, i.e. a conic with the same focus and directrix as the polar reciprocal of the circle  $MPP_1$ .

As in the figure  $F$  lies outside the circle to which  $PP_1$  is a tangent, the locus of  $Q$  is a hyperbola, the vertices of which are the poles of the tangents at  $A$  and  $A_1$ , the ends of the diameter through the origin  $F$ .

#### EXAMPLES ON CHAPTER VII.

Below are given in parallel columns some examples of reciprocal theorems :

1. The angles in the same segment of a circle are equal.

If a moveable tangent of a conic meet two fixed tangents, the intercepted portion subtends a constant angle at the focus.

2. Two of the common tangents of two equal circles are parallel.

If two conics have the same focus, and equal latera recta, the straight line joining two of their common points passes through the focus.

3. If a circle be inscribed in a triangle, the lines joining the vertices with the points of contact meet in a point.

If a triangle be inscribed in a conic, the tangents at the vertices meet the opposite sides in three points lying in a straight line.

4. If two chords be drawn from a fixed point on a circle at right angles to each other, the line joining their ends passes through the centre.

If two tangents of a conic move so that the intercepted portion of a fixed tangent subtends a right angle at the focus, the two moveable tangents meet in the directrix.

5. Any two tangents to a circle make equal angles with their chord of contact.

The line drawn from the focus to the intersection of two tangents bisects the angle subtended at the focus by their chord of contact.

6. If two chords at right angles to each other be drawn through a fixed point on a circle, the line joining their extremities passes through the centre.

The locus of the intersection of tangents to a parabola which cut at right angles is the directrix.

[Take the fixed point on the circle as the centre of the auxiliary circle, and the circle reciprocates into a parabola.]

7. The envelope of a chord of a circle which subtends a given angle at a given point on the circle is a concentric circle.

The locus of the intersection of tangents to a parabola, which cut at a given angle, is a conic having the same focus and the same directrix.

8. The rectangle under the segments of any chord of a circle through a fixed point is constant.

The rectangle under the perpendiculars let fall from the focus on two parallel tangents is constant.

[Take the fixed point as centre of auxiliary circle.]

9. If lines be drawn from the end of a diameter of a circle making equal angles with a fixed straight line in the plane of the circle, the chords subtended by these lines are parallel.

The points of intersection of tangents to a parabola, which are equally inclined to a given straight line, lie on a fixed straight line passing through the focus.

10. The portion of any tangent to a circle intercepted between two parallel tangents subtends a right angle at the centre.

The portion of the directrix intercepted between chords drawn from the ends of any focal chord of a conic to any point of the curve subtends a right angle at the focus.

11. Shew that the polar reciprocal of a parabola with respect to a circle having any point ( $S$ ) of the directrix as centre is an equilateral hyperbola.

[Draw the tangents to the parabola from the point  $S$ , which will be at right angles to each other since  $S$  is on the directrix. The reciprocals of their points of contact will be asymptotes to the reciprocal curve, because their points of contact (the poles of the tangents) are at an infinite distance. The tangents at the vertices can easily be drawn, since they are the polars of the points in which a line through  $S$  parallel to the bisector of the angle between the asymptotes meets the parabola.]

12. Given three points  $A$ ,  $B$  and  $C$ , on a parabola, and a point  $L$  on the directrix, draw the curve.

[If the three points are reciprocated with respect to a circle described with centre  $L$ , and a rectangular hyperbola described

passing through  $L$  and having the polars of  $A$ ,  $B$  and  $C$  for tangents (Ex.  $p$ , p. 194), any point on the hyperbola when reciprocated with respect to the same circle becomes a tangent to the parabola.]

13. If a conic be inscribed in a quadrilateral, shew that the angles subtended at a focus by the pairs of opposite sides are together equal to two right angles.

[Reciprocate the well-known theorem: The opposite angles of any quadrilateral inscribed in a circle are equal to two right angles.]

14. With the centre of perpendiculars of a triangle as focus are described two conics, one touching the sides and the other passing through the feet of the perpendiculars; prove that these conics will touch each other, and that their point of contact will lie on the conic which touches the sides of the triangle at the feet of the perpendiculars.

15. An hyperbola is its own reciprocal with respect to either circle which touches both branches of the hyperbola and intercepts on the transverse axis a length equal to the conjugate axis.

## CHAPTER VIII.

### ANHARMONIC RATIO AND ANHARMONIC PROPERTIES OF CONICS.

*Application of the signs + and - to determine the direction of segments of a right line.*

If  $A, B$  are two points in a straight line and it is necessary to discriminate whether the length  $AB$  is to be measured from  $A$  to  $B$  or from  $B$  to  $A$ , it may be done by calling the one direction positive and the other negative, the starting point in each case being called the origin.

Regard being paid to this convention we may evidently say

$$AB = -BA \text{ or } AB + BA = 0,$$

and an obvious interpretation of this equation is that if we go from  $A$  to  $B$  and back again from  $B$  to  $A$  we are finally at zero distance from the starting point.

The same thing is evidently true of any number of segments; for if we take three points  $A, B, C$  in any order in a straight line and travel from  $A$  to  $B$ , then from  $B$  to  $C$ , and finally from  $C$  to  $A$ , we arrive at the point we started from, and really perform the operation expressed by the equation

$$AB + BC + CA = 0.$$

Since  $-CA = AC$ , this may also be written

$$AB + BC = AC.$$

When the position of a point  $A$  is determined by its distance from an origin  $O$ , if we wish to refer it to another origin  $O_1$  anywhere on the line through  $O$  and  $A$ , we can always take

$$OA = O_1A - O_1O,$$

for this is identical with the equation

$$OA - O_1A + O_1O = 0,$$

and, since  $-O_1A = AO_1$ , with

$$OA + AO_1 + O_1O = 0.$$

*The difference of two segments OA, OB of a straight line with a common origin O is always equal to BA, whatever may be the magnitudes and directions of the segments.*

For the equation  $OA - OB = BA$

is identical with  $OA + AB + BO = 0$ .

*If a is the middle point of a segment Aa and M any point on the line through Aa,*

$$Ma = \frac{MA + Ma}{2} \text{ and } MA \cdot Ma = \overline{Ma}^2 - \overline{aa}^2,$$

for between the three points M, a, A the relation holds

$$Ma + aA + AM = 0,$$

$$Ma + aA = MA.$$

Similarly

$$Ma + aa = Ma;$$

therefore, adding these equations and remembering that  $aA = -aa$ , since a is midway between A and a,

$$Ma = \frac{MA + Ma}{2};$$

also, multiplying together the right and left-hand members, we get

$$\begin{aligned} MA \cdot Ma &= \overline{Ma}^2 + Ma(aA + aa) + aA \cdot aa \\ &= \overline{Ma}^2 - \overline{aa}^2, \end{aligned}$$

since  $aA = -aa$ .

Let A, B, C, D be four points in a straight line, then the ratio of the distances of one point A from two others B and D, divided by the ratio of the distances of the remaining point C from the same two (B and D), is called the Anharmonic Ratio of the range A, B, C, D; i.e. the anharmonic ratio of the range ABCD is the numerical value of the expression  $\frac{AB}{AD} \div \frac{CB}{CD}$ , which may also be

written  $\frac{AB}{AD} \cdot \frac{CD}{CB}$  or  $AB \cdot CD : CB \cdot AD$ . The sign of the ratio will depend on the signs of the segments of which it is composed, those which are measured in one direction being considered positive and those measured in the opposite direction negative.

Thus, if the four points are in the order from left to right  $ABCD$ , the three terms  $AB$ ,  $CD$  and  $AD$  in the above ratio are positive and the term  $CB$  is negative; and the ratio itself is negative.

Since four points in a straight line taken in pairs give six segments, there are really six anharmonic ratios corresponding to any range, three of which however are merely the inverse values of the remaining three. Thus instead of taking the ratio of the distances of  $A$  from  $B$  and  $D$  and dividing by the ratio of the distances of  $C$  from  $B$  and  $D$ , we might take the ratio of the distances of  $A$  from  $C$  and  $D$  and divide by the ratio of the distances of  $B$  from  $C$  and  $D$ , giving the expression

$$\frac{AC}{AD} \div \frac{BC}{BD}, \text{ or } \frac{AC}{AD} \cdot \frac{BD}{BC},$$

and in this case if the points are in the order  $ABCD$  all the segments are of the same sign and the ratio is positive.

Again, we might take the ratio of the distances of  $A$  from  $B$  and  $C$  and divide by the ratio of the distances of  $D$  from  $B$  and  $C$ , giving the expression

$$\frac{AB}{AC} \div \frac{DB}{DC}, \text{ or } \frac{AB}{AC} \cdot \frac{DC}{DB},$$

where two of the segments are of the same sign and two of opposite sign, so that the ratio is again positive.

In the above ratios the same point  $A$  has been associated successively with the three remaining  $C$ ,  $B$ ,  $D$ . In the first  $A$  and  $C$  may be said to be conjugate points, in the second  $A$  and  $B$ , and in the third  $A$  and  $D$ .

Of the three fundamental ratios formed as above, two are always positive and one negative, whatever the order of the points.

Besides these there are the three inverse ratios

$$\frac{AD}{AB} \div \frac{CD}{CB}, \quad \frac{AD}{AC} \div \frac{BD}{BC}, \quad \frac{AC}{AB} \div \frac{DC}{DB}.$$

It is of course necessary to retain throughout any investigation the particular order adopted at its commencement.

The anharmonic ratio of the range  $A, B, C, D$  is denoted by  $\{ABCD\}$ .

If the anharmonic ratio of a range  $= -1$ , the segments are in harmonic progression; for if the points occur in the order  $ACBD$  and  $\frac{AC}{AD} \div \frac{BC}{BD} = -1$ , we have

$$\frac{AC}{AD} = -\frac{BC}{BD} = \frac{CB}{BD},$$

since  $BC$  and  $CB$  are measured in opposite directions, and therefore of course the three segments  $AC, AB, AD$  are such that the first : the third :: difference between first and second : difference between second and third. If one of the points ( $D$  suppose) is at an infinite distance, the anharmonic ratio  $\frac{AC}{AD} \div \frac{BC}{BD}$  reduces to the simple ratio  $\frac{AC}{BC}$ , for it may be written  $\frac{AC}{BC} \div \frac{AD}{BD}$ , and  $AD$  is ultimately equal to  $BD$ .

PROP. 1. *If four fixed straight lines which meet in  $O$  be cut by any transversal in the points  $A, B, C, D$  (fig. 118), then will  $\{ABCD\}$  be constant.*

Draw the straight line  $aBc$  parallel to  $OD$ , and meeting  $OA, OC$  in  $a$  and  $c$ .

Then  $AB : AD :: aB : OD$  by similar triangles.

Similarly,  $CD : CB :: OD : cB$  „ „

therefore  $AB \cdot CD : AD \cdot CB :: aB : cB$ ,

or 
$$\frac{AB}{AD} \div \frac{CB}{CD} = \frac{aB}{cB};$$

but  $\frac{aB}{cB}$  is a constant ratio for all positions of  $ac$  parallel to  $OD$ ,

therefore  $\frac{AB}{AD} \div \frac{CB}{CD}$ , which is the anharmonic ratio of the four points  $A, B, C, D$ , is also constant.

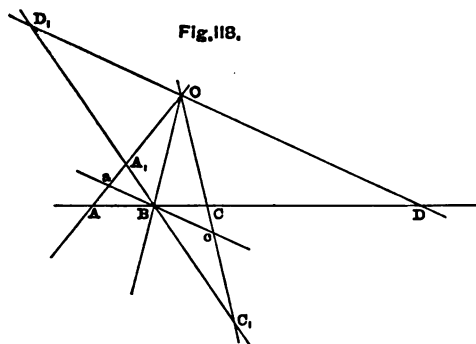


Fig. 118.

**DEF.** A bundle of lines drawn through one point is called a *pencil of rays*, or shortly a *pencil*.

The *anharmonic ratio of a pencil* of four rays is the anharmonic ratio of the range in which its rays are intersected by any transversal.

Pencils and ranges are said to be *equal* when their anharmonic ratios are equal, corresponding lines and points being taken for the comparison.

Equiangular pencils are evidently equal.

The anharmonic ratio of a pencil is denoted by  $O\{ABCD\}$ ;  $O$  being the vertex and  $OA, OB, OC, OD$  the rays of the pencil.

**PROP. 2.** *The transversal may cut the rays of the pencil on either side of the vertex.*

For if, in the preceding article a transversal is drawn through  $B$ , cutting  $OA$  in  $A_1$ ,  $OC$  in  $C_1$ , and  $OD$  in  $D_1$ , where  $D_1$  lies in  $DO$  produced, then, exactly as before,

$$A_1B : A_1D_1 :: aB : OD_1,$$

and

$$C_1D_1 : C_1B :: OD_1 : cB;$$

therefore

$$\frac{A_1B}{A_1D_1} \div \frac{C_1B}{C_1D_1} = \frac{aB}{cB} = \frac{AB}{AD} \div \frac{CB}{CD}.$$



If a transversal meet  $AO$  produced in  $\alpha$ ,  $BO$  produced in  $\beta$ ,  $CO$  produced in  $\gamma$ , and  $DO$  in  $\delta$ ,

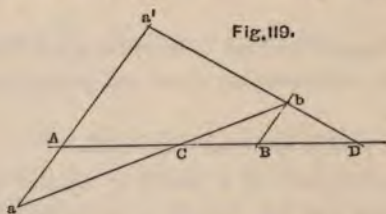
$$\{ABCD\} = \{\alpha\beta\gamma\delta\}.$$

If a transversal be drawn parallel to one of the rays of the pencil ( $OC$ ) suppose, meeting the other rays in  $a, b, d$ , we have as before for the anharmonic ratio of the range  $abcd$ , where  $c$  is at an infinite distance,  $\frac{ab}{ad}$ , which is therefore  $= \frac{AB}{AD} \div \frac{CB}{CD}$ .

If the pencil is harmonic  $\frac{ab}{ad} = -1$ ; therefore  $ab = -ad$ , or  $a$  is the centre point of  $bd$ .

PROBLEM 120. *Given the anharmonic ratio  $\lambda$  of four points, three of which are given in position, to determine the fourth (Fig. 119).*

Suppose that the three points  $A, C$ , and  $D$  of the anharmonic ratio  $\frac{AC}{AD} \div \frac{BC}{BD} = \lambda$  are given and the point  $B$  required. Through  $A$  draw any line and on it set off from  $A$  segments  $Aa, Aa'$  which



are to each other in the ratio  $\lambda$ : the segments must be taken on the same side of  $A$  if  $\lambda$  is positive, and on opposite sides (as in figure) if it is negative. Draw  $aC$  and  $a'D$  meeting in  $b$ , and draw  $bB$  parallel to  $Aa$  meeting the line  $ACD$  in  $B$ , which will be the required fourth point of the range; for by similar triangles we have

$$\frac{AC}{BC} = \frac{Aa}{Bb}, \quad \text{and} \quad \frac{AD}{BD} = \frac{Aa'}{Bb};$$

$$\therefore \frac{AC}{AD} \div \frac{BC}{BD} = \frac{Aa}{Aa'} = \lambda,$$

the ratio being negative, since  $AC$ ,  $AD$  and  $BD$  are measured in one direction and  $BC$  in the opposite.

The same construction determines the points  $C$  or  $D$  if the points  $A$ ,  $B$ ,  $D$  or  $A$ ,  $B$ ,  $C$  and the ratio  $\frac{AC}{AD} \div \frac{BC}{BD} = \lambda$  are given. To determine  $C$ , e.g. draw  $Bb$  parallel to  $Aa$  meeting  $a'D$  in  $b$ , and draw  $ab$  to meet  $AB$  in  $C$ .

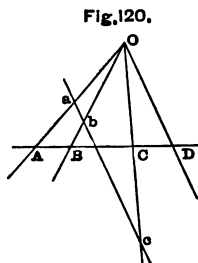
If the points  $B$ ,  $C$ ,  $D$  are given, the given anharmonic ratio may be written  $\frac{BD}{BC} \div \frac{AD}{AC} = \lambda$ , and the construction may be made by substituting the point  $B$  for the point  $A$ ; by drawing, i.e. through the point  $B$  any line and setting off on it from  $B$  segments  $Bb'$ ,  $Bb$  in the ratio  $\lambda$ , and joining  $b'$  to  $D$  and  $b$  to  $C$ .

If  $\lambda$ , instead of being a number positive or negative, is the ratio of two lines of given length, these lengths may themselves be set off from  $A$  to  $a$  and  $a'$ , on the same or on opposite sides of  $A$  according as the ratio is positive or negative.

PROP. 3. *If  $A$ ,  $B$ ,  $C$ ,  $D$  are four points in a straight line, then  $AB \cdot CD + AC \cdot DB + AD \cdot BC = 0$ , the general rule of signs being observed (Fig. 120).*

Divide by  $AB \cdot CD$ , and the equation becomes

$$\frac{AC}{AB} \div \frac{DC}{DB} + \frac{AD}{AB} \div \frac{CD}{CB} = 1.$$



Draw the lines  $OA$ ,  $OB$ ,  $OC$ ,  $OD$ ,  $O$  being any point, and draw a transversal parallel to  $OD$  meeting  $OA$ ,  $OB$ ,  $OC$  in  $a$ ,  $b$ ,

then 
$$\frac{AC}{AB} \div \frac{DC}{DB} = \frac{ac}{ab},$$

and 
$$\frac{BC}{BA} \div \frac{DC}{DA} = \frac{bc}{ba} = \frac{AD}{AB} \div \frac{CD}{CB},$$

so that the above equation may be written

$$\frac{ac}{ab} + \frac{cb}{ab} = 1,$$

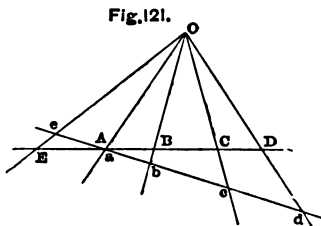
or 
$$ac + cb + ba = 0,$$

which is always true.

The above equation is formed by multiplying each term of the identity  $BC + CD + DB = 0$  by the distance of  $A$  from the remaining point, the first term  $BC$  by  $AD$ , the second  $CD$  by  $AB$ , and the third  $DB$  by  $AC$ .

PROP. 4. *If the anharmonic ratios of two systems of four points  $A, B, C, D$  and  $a, b, c, d$  taken on two straight lines and corresponding each to each are equal, and the lines are so placed that two homologous points  $A$  and  $a$  coincide, the three straight lines joining the remaining pairs of homologous points will meet in a point (Fig. 121).*

For if not, let  $Bb$  and  $Cc$  meet in  $O$ , and let  $OD$  meet the line  $ac$  in  $d_1$ : the pencil  $O, ABCD$  is met by two transversals  $AD$



and  $ad_1$ , and therefore the anharmonic ratio

$$\frac{AB}{AC} \div \frac{DB}{DC} = \frac{ab}{ac} \div \frac{d_1b}{d_1c};$$

but by hypothesis

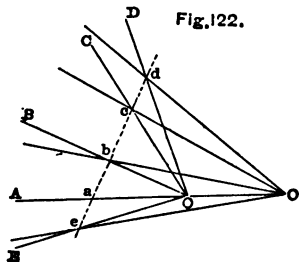
$$\frac{AB}{AC} \div \frac{DB}{DC} = \frac{ab}{ac} \div \frac{db}{dc},$$

therefore  $\frac{db}{dc} = \frac{d_1b}{d_1c}$ , which is impossible unless  $d$  and  $d_1$  coincide.

Two lines divided so that the anharmonic ratio of any four points on the one is equal to the anharmonic ratio of the four corresponding points on the other are said to be divided *homographically*.

PROP. 5. *If the anharmonic ratios of two pencils of four rays, corresponding each to each, are equal, and the pencils are so placed that two corresponding lines coincide in direction, the three points of intersection of the remaining homologous rays lie on a straight line (Fig. 122).*

Let  $O$  and  $O_1$  be the vertices of the pencils, the ray  $OA$  of the one coinciding in direction with the ray  $O_1a$  of the other, and let the



homologous rays  $OB$ ,  $O_1b$  meet in  $b$ , and  $OC$ ,  $O_1c$  in  $c$ ; the straight line  $bc$  will pass through the point  $d$  where  $OD$  intersects  $O_1d$ , for if not, let it meet  $OD$  in  $D$  and  $O_1d$  in  $d_1$ ; then, since the anharmonic ratios of the two pencils are equal, we have

$$\frac{ab}{ac} \div \frac{Db}{Dc} = \frac{ab}{ac} \div \frac{d_1b}{d_1c},$$

which is impossible unless  $d_1$  and  $D$  coincide.

Two pencils such that the anharmonic ratio of any four rays of the one is equal to the anharmonic ratio of the corresponding four rays of the other are said to be *homographic*.

PROBLEM 121. *Given any number of points  $A, B, C, D, E...$  on a straight line, and any three corresponding points (as  $a, b, c$ ) on a second line, to complete the homographic division of the second line (Fig. 121).*

Place the lines at any angle with each other and with two corresponding points (as  $A$  and  $a$ ) coinciding. Let the lines joining the remaining pairs of corresponding points ( $Bb$  and  $Cc$ ) meet in  $O$ , then the lines  $OD, OE...$  &c. will meet the second line in  $d, e, \&c.$ , the required points of homographic division.

The construction is obvious from the known property of the transversals of a pencil of rays.

It follows conversely that if a pencil of rays intersect any two lines in points  $B, b; C, c, \dots$  the lines are divided homographically, and that the point  $A$  in which the two lines intersect is its own homologue in both divisions.

PROBLEM 122. *Given a pencil of rays  $O.ABCDE...$  and any three corresponding rays  $O_1.abc$  of a second pencil, to complete the second pencil so that the two shall be homographic (Fig. 122).*

Place the pencils so that two corresponding rays ( $OA, O_1a$  suppose) shall be coincident in direction, let the homologous rays intersect in  $b$  and  $c$ ; the straight line  $bc$  will intersect the remaining rays of the given pencil in points on the required completing rays of the second. The construction is obvious from the known property of a transversal.

Conversely, when the corresponding rays of two pencils intersect in points on a straight line, the pencils are homographic, and the line  $OO_1$  joining the centres is common to both pencils and is coincident with its own homologue.

PROP. 6. *If  $A, B, C, D$  are four points on the circumference of a circle, and from any point  $O$  on the circumference the pencil  $O.ABCD$  is drawn, the anharmonic ratio of this pencil is constant.*

For if  $O_1$  is any other point on the circumference the pencil  $O_1.ABCD$  is equiangular with the pencil  $O.ABCD$ .

PROP. 7. *If four fixed tangents to a circle are met by any variable tangent in  $A, B, C, D$ , the anharmonic ratio of this range is constant.*

For the angles which the four points subtend at the centre are constant, and therefore the ranges are transversals of equiangular pencils.

If we reciprocate these theorems (Prob. 119), the four fixed points in the first correspond to four fixed tangents to a conic, the variable point  $O$  corresponds to a variable tangent, the lines  $OA, OB$ , &c. correspond to the points  $a, b, c, d$  in which the variable tangent cuts the fixed tangents; and since points corresponding to lines lie on lines through the centre of the auxiliary circle perpendicular to the lines to which they correspond, the pencil formed by joining  $a, b, c, d$  to the centre of the auxiliary circle is equiangular with the pencil  $O.ABCD$ , i.e. the anharmonic ratio of the range  $abcd$  is constant.

PROP. 8. *The reciprocal theorem to Prop. 6 therefore is "The anharmonic ratio of the points in which four fixed tangents to a conic cut any variable tangent is constant."*

PROP. 9. *By exactly similar reasoning the reciprocal theorem to Prop. 7 is "The anharmonic ratio of the pencil formed by joining any four fixed points on a conic to a variable fifth is constant."*

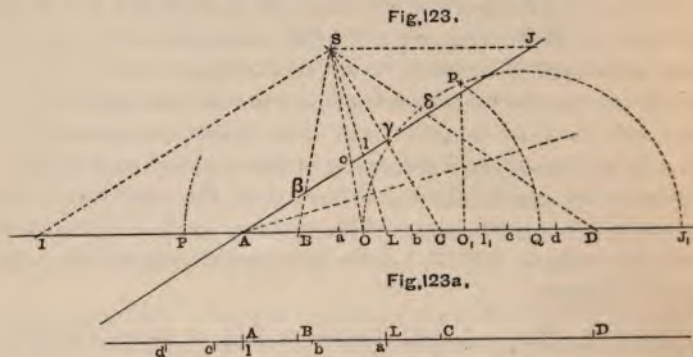
If the reciprocal figures be drawn, by observing the angles which correspond to the constant angles in the circle, it will be seen that the angles which the four points of the variable tangent in the first theorem subtend at either focus are constant; and that the angles are constant which are subtended at the focus by the four points in which the inscribed pencil meets the directrix in the second theorem.

#### HOMOGRAPHIC RANGES IN THE SAME STRAIGHT LINE. DOUBLE POINTS.

When two lines divided homographically are superposed, there exist, in general, two points, each of which considered as belonging to the first division coincides with its homologue in the

second division. They may be called *double points*, since each represents two coincident homologous points. The double points may become imaginary.

Let  $A, B, C, D...$  (fig. 123) be any points on a straight line,  $S$  any point, and  $AJ$  a line through  $A$  making any angle with  $AD$ , and meeting  $SB, SC$ , &c. in  $\beta, \gamma...$  respectively.



The ranges  $ABCD...$  and  $A\beta\gamma\delta...$  are of course homographic, and if the second range be rotated round  $A$  till it coincides in direction with  $AD$ , the point  $A$ , considered as belonging to the first division, will evidently coincide with its own homologue in the second division, as will also the point  $L$  determined by drawing  $SL$  perpendicular to the line bisecting the angle  $DAJ$ .

Two homographic ranges in the same straight line formed as above possess therefore two double points.

Instead however of the two ranges being formed merely by the rotation of the second about  $A$ , the second may in addition be moved along  $AD$  into any position to the right or left, bringing (as in the figure) the point corresponding to  $A$  to  $a$ ,  $\beta$  to  $b$ ,  $\gamma$  to  $c...$  In this case also two double points in general exist, which may be thus found:—

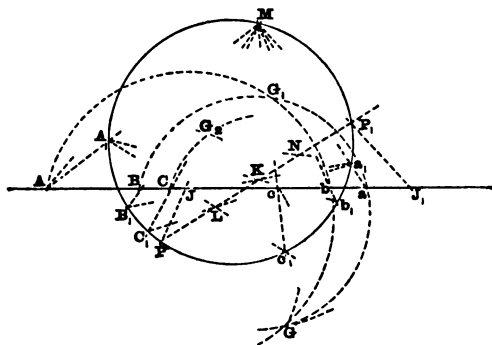
**PROBLEM 123.** *Given two homographic ranges  $ABCD...$ ,  $abcd...$  in the same straight line, to determine the double points (Fig. 124).*

*Draw any circle whatever, and from any point  $M$  on it draw*



$MA, MB, MC$  cutting the circumference in  $A_1, B_1, C_1$ , and draw  $Ma, Mb, Mc$  cutting the circumference in  $a_1, b_1, c_1$ .

Fig. 124.



Draw  $A_1b_1, B_1a_1$  intersecting in  $K$ , and  $A_1c_1, C_1a_1$  intersecting in  $L$ , and draw  $KL$  cutting the circumference in  $P$  and  $P_1$ . The lines  $MP, MP_1$  will cut  $Aa$  in the required double points  $J$  and  $J_1$ .

For  $\{ABCJ\}$

$$\begin{aligned} &= M\{A_1B_1C_1P\} = a_1\{A_1B_1C_1P\} \\ &= \{NKL P\}, \end{aligned} \quad (\text{Prop. 6.})$$

where  $N$  is the point of intersection of  $KL$  and  $A_1a_1$

$$\begin{aligned} \text{and} \quad \{abcJ\} &= M\{a_1b_1c_1P\} = A_1\{a_1b_1c_1P\} \\ &= \{NKL P\}, \end{aligned}$$

i.e. the point  $J$ , considered as belonging to the first range, coincides with its own homologue in the second and is therefore a double point. Similarly for the point  $J_1$ .

It will be seen that the line  $PP_1$  is a Pascal line (Prob. 86) in the circle, for  $C_1b_1$  and  $B_1c_1$  intersect on it.

The double points may also be determined thus:—On  $Ab, Ba$  as chords describe two circles passing also through any arbitrary point  $G$  and intersecting again in  $G_1$ , on  $Ac, Ca$  as chords describe circles passing through  $G$  and intersecting again



in  $G_2$ . The circle through  $G, G_1, G_2$  passes also through the required double points of the ranges.

A third construction for the double points is shewn in fig. 123. Through  $A$ , any point of the first range, draw  $AJ$ , a line making any angle with the given line. On it make  $A\beta = ab, A\gamma = ac$ , where  $a, b, c$  are the points of the second range corresponding to the points  $A, B, C$  of the first. Draw  $B\beta, C\gamma$  intersecting in  $S$ . Through  $S$  draw  $SJ$  parallel to  $AC$  meeting  $AJ$  in  $J$ , so that  $J$  is the point on the range  $A\beta\gamma \dots$  corresponding to an infinitely distant point on the range  $ABC \dots$ ; and draw  $SI$  parallel to  $AJ$ , so that  $I$  is the point on the range  $ABC \dots$  corresponding to an infinitely distant point on the range  $A\beta\gamma \dots$ . Make  $aJ_1$  on the superposed ranges  $= AJ$ , so that the range  $A\beta\gamma \dots J$  is identical with the range  $abc \dots J_1$ . Bisect  $IJ_1$  in  $O$ , which will be equidistant from the required double points. Find the point  $O_1$  on the range  $abc \dots$  corresponding to the point  $O$  considered as belonging to the range  $ABC \dots$  [i.e. join  $SO$  cutting  $AJ$  in  $o$ , and make  $aO_1 = Ao$ ]. Then the mean proportional between  $OO_1$  and  $OJ_1$  will be the distance from  $O$  of the required double points  $P$  and  $Q$ .

On page 17 a system of pairs of points on a straight line such that  $XA \cdot Xa = XB \cdot Xb = XC \cdot Xc = \dots = XP^2 = XQ^2$  was defined as a system in Involution, any two corresponding points such as  $A, a$  being called conjugate points, the point  $X$  the centre, and the points  $P$  and  $Q$  the foci of the involution.

PROP. 10. *When three pairs of conjugate points are in involution, the anharmonic ratio of any four of the points is equal to the anharmonic ratio of their four conjugates, i.e. taking any four  $A, B, C, a$  and their four conjugates  $a, b, c, A$ ,*

$$\frac{AB}{AC} \div \frac{aB}{aC} = \frac{ab}{ac} \div \frac{Ab}{Ac};$$

for if  $O$  is the centre of the involution

$$OA : OB :: Ob : Oa,$$

$$\therefore OA - OB : OB :: Ob - Oa : Oa,$$

or

$$AB : ab :: OB : Oa,$$

and

$$OA + Ob : Ob :: OB + Oa : Oa ;$$

$$\therefore Ab : aB :: Ob : Oa,$$

$$\therefore \frac{AB \cdot Ab}{ab \cdot aB} = \frac{OB \cdot Ob}{Oa^2} = \frac{OA}{Oa}.$$

Similarly of course  $\frac{AC \cdot Ac}{ac \cdot aC} = \frac{OA}{Oa} = \frac{AB \cdot Ab}{ab \cdot aB},$

$$\therefore \frac{AB \cdot Ab}{AC \cdot Ac} = \frac{ab \cdot aB}{ab \cdot aC},$$

which may be written

$$\frac{AB}{AC} \div \frac{aB}{aC} = \frac{ab}{ac} \div \frac{Ab}{Ac},$$

which proves the proposition.

A series of points in involution consists of two homographic ranges, the directions of which coincide, and in which to any point whatever  $M$  of the line the same point  $m$  corresponds, whether  $M$  be considered as belonging to the first or second system.

For consider  $M$  as belonging to the range  $ABC\dots$ ,

and  $m$  „ „ „  $abc\dots$ ;

then, since the ranges are homographic,

$$\frac{MA}{MB} \div \frac{CA}{CB} = \frac{ma}{mb} \div \frac{ca}{cb}.$$

If they are also in involution we must be able to interchange  $m$  and  $M$ , i.e. considering  $M$  as belonging to the range  $abc\dots$  and  $m$  as belonging to the range  $ABC\dots$ , we must have

$$\frac{Ma}{Mb} \div \frac{ca}{cb} = \frac{mA}{mB} \div \frac{CA}{CB}.$$

Dividing each term of the first equation by the opposing term of the second,

$$\frac{MA}{MB} \div \frac{mA}{mB} = \frac{ma}{mb} \div \frac{Ma}{Mb},$$

i.e. the anharmonic ratio of the four points  $M, A, B, m$  is equal to the anharmonic ratio of their four conjugates  $m, a, b, M$ , or the points are in involution.

PROP. 11. *It is always possible to superimpose two homographic ranges, so that the two divisions shall be in involution.*

For it has been shewn (p. 220) that two pairs of corresponding points can be found equidistant from each other, by drawing viz. through  $S$  (fig. 123) a line  $Ll$  perpendicular to the line bisecting the angle between the ranges when the second is placed at any angle with the first and with two corresponding points  $A, a$  at the intersection: the pairs of points  $A, L$  and  $a, l$  are then equidistant. If now the two ranges are superimposed with the point  $a$  coinciding with the point  $L$ , and the point  $l$  coinciding with the point  $A$  (fig. 123a), the two ranges will be in involution.

The foci (p. 17) are points at which pairs of conjugate points coincide, and their existence is only possible when the points of any conjugate pair in the involution are both on the same side of the centre.

Thus if the points are in the order  $ABab$ , the centre  $X$  must fall between  $B$  and  $a$  in order that the products  $XA.Xa$  and  $XB.Xb$  may have the same sign, and that sign will be negative since the segments in each of the products are measured in opposite directions; but a square number is always positive, and therefore no foci exist.

If three segments  $Aa, Bb, Cc$  are in involution and one overlaps (as above) another, i.e. if the points are in the order  $ABab$ , it will also overlap the third. This is evident if we consider that if  $C$  lies to the left of  $X$ , and  $XC$  is greater than  $XA$ ,  $c$  must be on the opposite side of  $X$ , and  $Xc$  must be less than  $Xa$  and vice versa, and similarly with regard to  $Bb$ .

Conversely, if the segment  $Aa$  does not overlap  $Bb$ , it cannot overlap  $Cc$ , nor can  $Bb$  and  $Cc$  overlap.

The centre  $X$  forms with any two pairs of points  $A, a$  and  $B, b$  an involution in which the conjugate to the centre is at an infinite distance, for if  $x$  is conjugate to  $X$ , the anharmonic ratio of the four points  $XABx$  = the anharmonic ratio of the four conjugate points  $xabX$ ,

$$\text{i.e. } \frac{XA}{XB} \div \frac{xA}{xB} = \frac{xa}{xb} \div \frac{Xa}{Xb};$$

but  $\frac{XA}{XB} = \frac{Xb}{Xa}$ , and therefore  $\frac{xB}{xA} = \frac{xa}{xb}$ ,

or  $xA \cdot xa = xB \cdot xb \dots \dots \dots (a).$

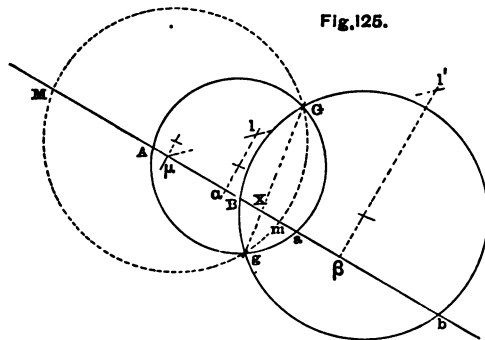
Now  $x$  cannot lie between  $b$  and  $A$  because the segment  $Xx$  must overlap both  $Aa$  and  $Bb$ , i.e.  $x$  must lie either to the right of  $b$  or to the left of  $A$ ; if it lies to the right of  $b$  the segment  $xa$  is greater than the segment  $xb$  and the segment  $xA$  is greater than the segment  $xB$ , and therefore the equation (a) cannot hold unless  $x$  is at an infinite distance, in which case the segments  $xA$ ,  $xa$ ,  $xB$ ,  $xb$  are ultimately equal.

Similarly  $x$  cannot lie to the left of  $A$  except at an infinite distance, and similar reasoning applies to points in the position shewn in figs. 12 and 13.

PROP. 12. *If on two segments  $Aa$ ,  $Bb$  of a right line as chords, any two arcs of circles are described, their common chord passes through the centre  $X$  of the involution  $A, a$  and  $B, b$  (Fig. 125).*

For  $XA \cdot Xa = XG \cdot Xg = XB \cdot Xb.$

Fig. 125.



If the segments overlap as in the above figure, they may be taken as diameters of the intersecting circles, and the chord  $Gg$  will be perpendicular to the line  $AaBb$ . If the points are situated as in figs. 12 and 13, circles described on  $Aa$  and  $Bb$  as diameters will not intersect in real points, but the centre  $X$  will lie on the radical axis of the two circles.



If a third circle be described passing through the points  $Gg$  and cutting  $Ab$  in the points  $M, m$  (fig. 125) it follows, since

$$XM \cdot Xm = XG \cdot Xg = XA \cdot Xa = XB \cdot Xb,$$

that  $M$  and  $m$  are another pair of conjugate points in the involution.

The same is evidently true of *any* line cutting the circumferences of all three circles, and we have the important proposition:—

*If three circles pass through two given points, any straight line meeting the circles does so in a series of points in involution, the two points on the same circle being conjugate.*

When the three circles described on three segments in involution as diameters intersect, straight lines drawn from either of the points of intersection to the ends of each segment are perpendicular to each other; it follows, that *when three segments of a straight line are in involution, two points (real or imaginary) exist, at each of which each segment subtends a right angle; and conversely, that if a right-angled triangle turns round that angular point as centre, the segments which it intercepts on a fixed right line in any three of its positions have their extremities in involution.*

PROBLEM 124. *Given  $A, a$  and  $B, b$ , two pairs of conjugate points, and  $C$  a fifth point of the involution, to determine  $c$  the point conjugate to  $C$ .*

Through any arbitrary point  $G$  describe segments of circles having  $Aa, Bb$  as chords; they will intersect in a second point  $g$ , and a circle described through the three points  $G, g$  and  $C$  will intersect  $AC$  in  $c$ , the required conjugate point.

Or thus:—Take any arbitrary point  $G$  and draw  $GA, GB, GC$ ; draw any triangle with its vertices on these lines and two of its sides passing through  $a, b$ . The remaining side will pass through  $c$ .

If the point  $C$  be at infinity, the same method will give us the centre of the system.

The construction for this case is, "Through  $A, B$  draw any pair of parallels  $Ah, Bk$ , and through  $a, b$  a different pair of parallels  $ah, bk$ ; then  $hk$  will pass through the centre of the system."

PROP. 13. *If  $Aa, Bb, Cc$  are any three fixed segments of a straight line, and  $\alpha, \beta, \gamma$  their centre points, and if  $m$  is any point on the line, the function  $mA \cdot ma \cdot \beta\gamma + mB \cdot mb \cdot \gamma\alpha + mC \cdot mc \cdot \alpha\beta$  is of constant value whatever the position of  $m$ , the general rule of signs being observed.*

Take  $M$  any other point on the line, then (p. 209)

$$mA = MA - Mm, \quad ma = Ma - Mm;$$

$$\begin{aligned} \therefore mA \cdot ma &= MA \cdot Ma - (MA + Ma) Mm + Mm^2 \\ &= MA \cdot Ma - 2Ma \cdot Mm + Mm^2, \end{aligned}$$

$$\text{and} \quad mB \cdot mb = MB \cdot Mb - 2M\beta \cdot Mm + Mm^2,$$

$$\text{and} \quad mC \cdot mc = MC \cdot Mc - 2M\gamma \cdot Mm + Mm^2.$$

Multiply these equations by  $\beta\gamma, \gamma\alpha$  and  $\alpha\beta$  respectively, then

$$\text{since (p. 209)} \quad \beta\gamma + \gamma\alpha + \alpha\beta = 0,$$

$$\text{and (prop. 3)} \quad Ma \cdot \beta\gamma + M\beta \cdot \gamma\alpha + M\gamma \cdot \alpha\beta = 0,$$

$$\begin{aligned} \text{we get} \quad mA \cdot ma \cdot \beta\gamma + mB \cdot mb \cdot \gamma\alpha + mC \cdot mc \cdot \alpha\beta \\ = MA \cdot Ma \cdot \beta\gamma + MB \cdot Mb \cdot \gamma\alpha + MC \cdot Mc \cdot \alpha\beta, \end{aligned}$$

which proves the proposition.

PROP. 14. *If three conjugate pairs of points  $A, a; B, b; C, c$  are in involution, and  $\alpha, \beta, \gamma$  the centre points of the segments  $Aa, Bb, Cc$ ; if any point  $m$  be taken on the same straight line,*

$$mA \cdot ma \cdot \beta\gamma + mB \cdot mb \cdot \gamma\alpha + mC \cdot mc \cdot \alpha\beta = 0,$$

*the general rule of signs being observed.*

By the last proposition the value of the expression is constant whether the points are in involution or no. When they are in involution the value is zero when  $m$  coincides with the centre of the involution, since then

$$mA \cdot ma = mB \cdot mb = mC \cdot mc,$$

so that the equation may then be written

$$mA \cdot ma (\beta\gamma + \gamma\alpha + \alpha\beta) = 0,$$

which is evidently true since (p. 209)

$$\beta\gamma + \gamma\alpha + \alpha\beta$$

is always zero; and this proves the proposition.

PROP. 15. *If  $A, a; B, b; C, c$  are three pairs of conjugate points in involution, and  $\alpha, \beta, \gamma$  the centre points of the segments  $Aa, Bb, Cc$ ,*

$$\frac{AB \cdot Ab}{AC \cdot Ac} = \frac{\alpha\beta}{\alpha\gamma} = \frac{aB \cdot ab}{aC \cdot ac},$$

*with similarly formed equations for the remaining points.*

In the equation proved in the last proposition, suppose that the point  $m$  coincides with the point  $A$ . The first term then becomes zero, since  $mA$  is zero, and the equation is

$$AB \cdot Ab \cdot \gamma\alpha + AC \cdot Ac \cdot \alpha\beta = 0;$$

or, since  $\gamma\alpha = -\alpha\gamma$ ,

$$\frac{AB \cdot Ab}{AC \cdot Ac} = \frac{\alpha\beta}{\alpha\gamma} = \frac{aB \cdot ab}{aC \cdot ac},$$

if we make  $m$  coincide with  $a$ .

PROBLEM 125. *Given two pairs of points  $A, a$  and  $B, b$  in a straight line, to find on the same line a fifth point such that the product of its distances from one pair shall be to the product of its distances from the other in a given ratio  $\lambda$ , i.e. given  $A, a$  and  $B, b$ , to determine a point  $M$  such that  $\frac{MA \cdot Ma}{MB \cdot Mb} = \lambda$  (Fig. 125).*

Take any arbitrary point  $G$  and describe circles passing through  $A, a, G$  and  $B, b, G$  and intersecting again in  $g$ . Their centres will of course lie on lines perpendicular to  $Aa, Bb$  bisecting these segments; let  $\alpha$  and  $\beta$  be the points of bisection. Divide  $\alpha\beta$  in the point  $\mu$ , so that  $\frac{\alpha\mu}{\beta\mu} = \lambda$ , i.e. on any parallel lines through  $\alpha$  and  $\beta$  make  $\alpha l$  = the numerator, and  $\beta l'$  = the denominator of the given ratio, the lengths  $\alpha l, \beta l'$  being set off on the same side of  $\alpha\beta$  if  $\lambda$  is positive, and on opposite sides if it is negative.

Describe a third circle to pass through  $G$  and  $g$ , and with its centre on a line through  $\mu$  perpendicular to  $Ab$ . This circle will cut  $Ab$  in two points  $M$  and  $m$ , either of which fulfils the required conditions of the problem.

*Proof.* The points  $Mm$  as found are evidently in involution with the points  $Aa$  and  $Bb$  (p. 226).

$$\therefore \frac{MA \cdot Ma}{MB \cdot Mb} = \frac{\mu a}{\mu \beta} = \frac{mA \cdot ma}{mB \cdot mb} \quad (\text{prop. 15});$$

but  $\frac{\mu a}{\mu \beta} = \lambda$  by construction,

which proves the construction.

If the segments  $Aa$ ,  $Bb$  overlap the problem is always possible, but otherwise cases of impossibility may arise owing to the position of the point  $\mu$ . The problem becomes impossible if  $\mu$  falls between  $P$  and  $Q$ , the foci of the involution  $A$ ,  $a$  and  $B$ ,  $b$ .

1. If one of the segments falls entirely within the other (as in fig. 12) their centre points  $a$  and  $\beta$  lie outside the segment  $PQ$  and both on the same side of it. If  $\mu$  falls within the segment  $PQ$  the ratio  $\frac{\mu a}{\mu \beta} = \lambda$  is positive, and its value lies between  $\frac{Pa}{P\beta}$  and  $\frac{Qa}{Q\beta}$ . In order that the problem may be possible in this case  $\lambda$  must be negative or must not be between these limits.

2. If the segments are entirely outside each other (as in fig. 13) their centre points lie outside the segment  $PQ$ , but on opposite sides of it. If  $\mu$  falls on  $PQ$  the ratio  $\frac{\mu a}{\mu \beta}$  is negative, and its absolute value is between  $\frac{Pa}{P\beta}$  and  $\frac{Qa}{Q\beta}$ . In order that the problem may be possible in this case  $\lambda$  must be positive or must not be between these limits.

**PROBLEM 126.** *Given two straight lines  $AL$ ,  $BL_1$ , and a fixed point on each  $A$  and  $B$ . Through a given point  $R$  to draw a straight line meeting  $AL$  in  $a$  and  $BL_1$  in  $b$ , so that the segments  $Aa$  and  $Bb$  shall be to each other in a given ratio  $\lambda$  (Fig. 126).*

Imagine two variable points  $a_1$ ,  $b_1$  to move along  $Aa$  and  $Bb$  respectively, so that in corresponding positions  $\frac{Aa_1}{Bb_1} = \lambda$ . The points in corresponding different positions would form homographic ranges





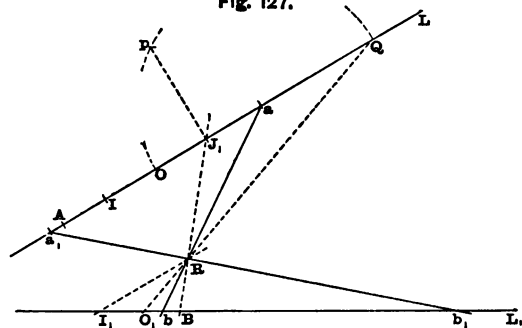
either side of the origin  $B$ ; so that if the directions in which the segments are measured is not prescribed, the problem admits of four solutions instead of two.

**PROBLEM 127.** *Given two straight lines  $AL$ ,  $BL_1$  and a fixed point  $A$ ,  $B$  on each, to draw through a given point  $R$  a line meeting  $AL$  in  $a$  and  $BL_1$  in  $b$ , so that the rectangle  $Aa \cdot Bb$  shall have a given value  $v$  (Fig. 127).*

Exactly as in the last problem, if points  $a_1, a_2, a_3, \dots$  are taken on  $AL$ , and points  $b_1, b_2, b_3, \dots$  on  $BL_1$ , connected by the relation

$$Aa_1 \cdot Bb_1 = Aa_2 \cdot Bb_2 = Aa_3 \cdot Bb_3 \dots = v,$$

Fig. 127.



these points will form homographic ranges; and if the pencil  $R \cdot b_1 b_2 b_3, \dots$  be drawn meeting  $AL$  in  $a_1, a_2, a_3, \dots$  respectively, the double points of the homographic ranges  $a_1, a_2, a_3, \dots$ ,  $a_1, a_2, a_3, \dots$  will be points on the lines required.

Corresponding to the point at infinity on the range  $a_1, a_2, a_3, \dots$  we must evidently have the point  $B$  on the range  $b_1, b_2, b_3, \dots$ , and therefore the line  $BR$  meets  $AL$  in the required point  $J_1$ . Corresponding to the point at infinity on the range  $a_1, a_2, a_3, \dots$  we have the point  $I_1$  on  $BL_1$  found by drawing  $RI_1$  parallel to  $AL$ , and then  $I$  is determined by making  $AI \cdot BI_1 = v$ . Bisect  $IJ_1$  in  $O$  and determine  $O_1$  on  $BL_1$ , so that  $AO \cdot BO_1 = v$ . Draw  $RO_1$  meeting  $AL$  in  $Q$ , and take a mean proportional ( $Op$ ) between  $OJ_1$  and  $OQ$ . Make  $Oa = -Oa_1 = Op$ , and  $Ra, Ra_1$  will be lines fulfilling the conditions of the problem.

PROBLEM 128. To draw a triangle having its vertices on three given lines and its sides passing through three given points (Fig. 128).

Let  $JJ_1$ ,  $KK_1$ ,  $LL_1$  be the given lines, and  $A$ ,  $B$ ,  $C$  the given points. Through one of the points, as  $A$ , draw any three lines

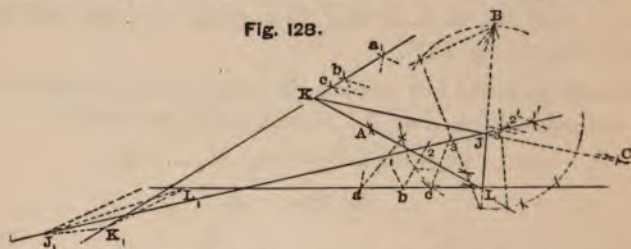


Fig. 128.

meeting two of the lines, as  $LL_1$ ,  $KK_1$ , in  $a$ ,  $b$ ,  $c$ ,  $a_1$ ,  $b_1$ ,  $c_1$  respectively. Draw the pencils  $B. abc$ ,  $C. a_1b_1c_1$  cutting the third line  $JJ_1$  in  $1$ ,  $2$ ,  $3$ ,  $1'$ ,  $2'$ ,  $3'$  respectively, which will evidently be homographic ranges. Find the double points  $JJ_1$  of the ranges (p. 221) and each will be the vertex of a triangle fulfilling the required condition, for

$$\{123J\} = \{abcL\} = \{1'2'3'J\} = \{a_1b_1c_1K\},$$

and since the rays  $aa_1$ ,  $bb_1$ ,  $cc_1$  pass through  $A$  so also must the ray  $KL$ .

In the figure  $JKL$ ,  $J_1K_1L_1$  are the two triangles.

PROP. 16. If a quadrilateral  $ABCD$  be inscribed in a conic, and any transversal be drawn meeting the four sides in  $a$ ,  $b$ ,  $c$ ,  $d$  and the conic in  $e$  and  $g$ , then the three pairs of points  $ac$ ,  $bd$ ,  $eg$  are in involution (Fig. 85.)

Let  $ABCD$  be the angles of the quadrilateral,  $AB$ ,  $DC$  meeting in  $E$ , and  $BC$ ,  $DA$  in  $F$ .

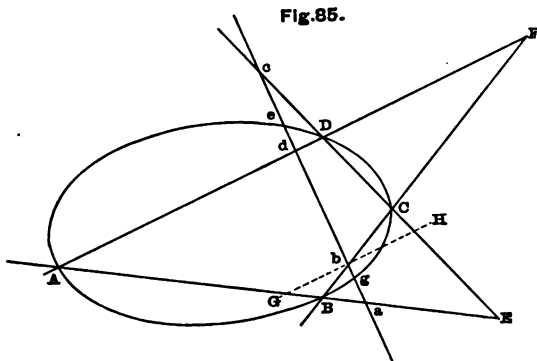
Let a transversal cut  $AB$  in  $a$ ,  $BC$  in  $b$ ,  $CD$  in  $c$ ,  $DA$  in  $d$ , and the curve in  $e$  and  $g$ .

The rectangles  $de \cdot dg$ ,  $dA \cdot dD$  are in the ratio of the squares on parallel diameters, as also are the rectangles  $be \cdot bg$  and  $bB \cdot bC$ ; but the squares on the diameters parallel to  $AD$  and  $BC$  are in the ratio

$$FA \cdot FD : FC \cdot FB;$$

$$\therefore \frac{de \cdot dg}{be \cdot bg} = \frac{dA \cdot dD}{bB \cdot bC} \cdot \frac{FB \cdot FC}{FA \cdot FD}.$$

Fig. 85.



But

$$\begin{aligned} \frac{dA}{da} &= \frac{\sin a}{\sin A}, \quad \frac{dD}{dc} = \frac{\sin c}{\sin D}, \\ \frac{bB}{ba} &= \frac{\sin a}{\sin B}, \quad \frac{bC}{bc} = \frac{\sin c}{\sin C}; \\ \therefore \frac{dA \cdot dD}{bB \cdot bC} &= \frac{da \cdot dc}{ba \cdot bc} \cdot \frac{\sin B \sin C}{\sin A \sin D} \\ &= \frac{da \cdot dc}{ba \cdot bc} \cdot \frac{FA \cdot FD}{FB \cdot FC}, \end{aligned}$$

i.e.

$$\frac{de \cdot dg}{be \cdot bg} = \frac{da \cdot dc}{ba \cdot bc},$$

or

$$\frac{de}{da} \div \frac{be}{ba} = \frac{bg}{bc} \div \frac{dg}{dc},$$

i.e. the anharmonic ratio of the four points  $d, e, a, b$  is equal to that of their four conjugates

$$b, g, c, d,$$

or the three pairs of points are in involution.

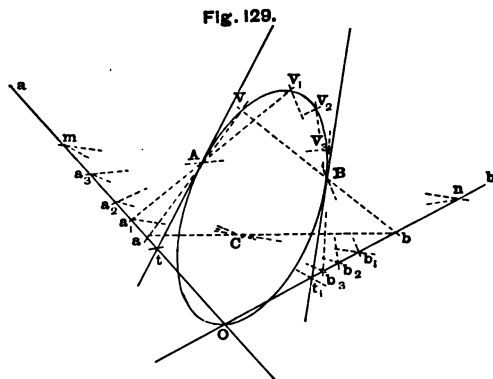
Since the diagonals of the quadrilateral form a particular case of a conic passing through the four points, it follows that the points in which the transversal cuts the diagonals are another pair in involution with  $ac, bd$ , &c.

**COR.** If the transversal be a tangent to the curve (meets it, that is, in two coincident points), it follows that the point of

contact is a focus of the involution formed by the three pairs of points in which the tangent cuts the opposite sides and diagonals.

*MacLaurin's method of generating conic sections :—*

Triangles are described, whose sides pass through three fixed points  $A, B, C$ , and whose base angles move on two fixed lines  $Oa, Ob$ : the vertices will lie on a conic section (fig. 129).



Suppose four such triangles drawn, then since the pencils through  $A$  and  $B$  are both homographic with the system through  $C$ , they are homographic with each other; therefore  $A, B, V, V_1, V_2, V_3$ , lie on the same conic section (Prop. 9). Now if the first three triangles be fixed, it is evident that the locus of  $V_3$  is the conic section passing through

$$A, B, O, V_1, V_2.$$

It follows of course that the locus of the intersection of homologous rays in two homographic pencils is a conic section.

*Newton's method of generating conic sections :—*

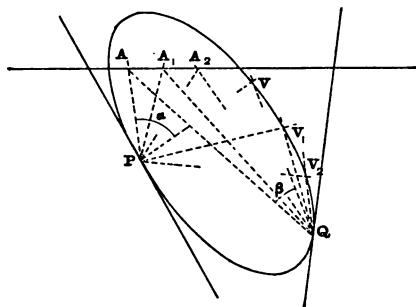
Two angles of constant magnitude move about fixed points  $P, Q$ ; the intersection of two of their sides traverses the right line  $AA_1$ ; then the locus of  $V$ , the intersection of their other two sides, will be a conic passing through  $P, Q$  (fig. 130).

Take four positions of the angles, then

$$P \cdot \{AA_1A_2A_3\} = Q \cdot \{AA_1A_2A_3\};$$

but  $P . \{AA_1A_2A_3\} = P . \{VV_1V_2V_3\}$   
 and  $Q . \{AA_1A_2A_3\} = Q . \{VV_1V_2V_3\}$ ,  
 since the angles of the pencils are the same;  
 $\therefore P . \{VV_1V_2V_3\} = Q . \{VV_1V_2V_3\}$ ,  
 and therefore, as before, the locus of  $V_3$  is a conic through  
 $P, Q, V, V_1, V_2$ .

Fig. 130.



*M. Chasles' extension of Newton's method.*

If the point  $A$  instead of moving on a right line moves on any conic passing through the points  $P, Q$ , the locus of  $V$  is still a conic section, since

$$P . \{AA_1A_2A_3\} = Q . \{AA_1A_2A_3\}.$$

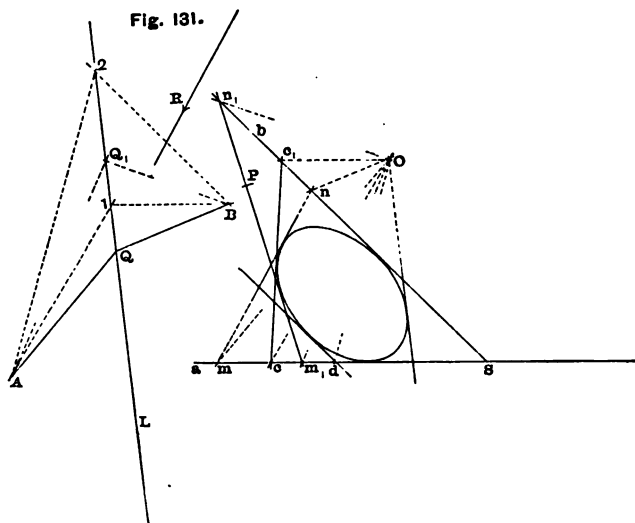
PROP. 17.\* *If there be any number of points  $a, b, c, d$ , &c. on a right line, and a homographic system  $a_1, b_1, c_1, d_1$ , &c. on another line, the lines joining corresponding points will envelope a conic.*

For if we construct the conic touched by the two given lines and by three lines  $aa_1, bb_1, cc_1$ , then, by the anharmonic property of the tangents of a conic (Prop. 8), any other of the lines  $dd_1$  must touch the same conic.

PROBLEM 129. *Given two points  $A$  and  $B$  and a line  $L$ ; given also two lines  $Sa, Sb$ , and a point  $O$ ; to find on  $L$  a point,  $Q$ , such that if  $Om, On$  be drawn parallel respectively to  $AQ$  and  $BQ$ , meeting  $Sa$  in  $m$  and  $Sb$  in  $n$ , the line  $mn$  shall*

- (a) be parallel to a given direction  $R$ ,
- (β) pass through a given point  $P$  (Fig. 131).

If two homographic pencils be drawn having  $O$  as common vertex, the envelope of the lines formed by joining the points in which the rays of the one meet  $Sa$  to the points in which the corresponding rays of the other meet  $Sb$ , is a conic section (Prop. 17).



Draw therefore any lines  $A1, A2, A3$ , &c., meeting  $L$  in  $1, 2, 3, \dots$  and through  $O$  draw parallels meeting  $Sa$  in  $c, d, e, \dots$  and parallels to  $B1, B2, B3, \dots$  meeting  $Sb$  in  $c_1, d_1, e_1, \dots$ . The pencils  $O(cde\dots)$ ,  $O(c_1d_1e_1\dots)$  are homographic because the pencils  $A(1, 2, 3\dots)$  and  $B(1, 2, 3\dots)$  are so, and therefore  $cc_1, dd_1, ee_1$  are tangents to a conic touching  $Sa$  and  $Sb$ , and which can therefore be constructed (Probs. 84 and 114).

In the figure the point  $d_1$  is at an infinite distance (i.e. the tangent  $dd_1$  is parallel to  $Sb$ ) because the line  $B2$  is drawn parallel to  $Sb$ .

A line through  $O$  parallel to the given line  $L$  is evidently a tangent to the conic to be constructed, so that it is only necessary to draw two pairs of lines  $A1, B1, A2, B2$ , since five tangents to the required conic are then known.

For the solution of the first part ( $\alpha$ ) draw  $mn$ , a tangent to this conic, parallel to the given direction  $R$ , and meeting  $Sa$  in  $m$  and  $Sn$  in  $n$ .

Then  $AQ$  drawn parallel to  $Om$  and  $BQ$  parallel to  $On$  will necessarily meet on the line  $L$  and determine the required point  $Q$ .

For the solution of the second part ( $\beta$ ) it is evidently only necessary to draw a tangent from  $P$  to the conic cutting  $Sa$  in  $m_1$  and  $Sb$  in  $n_1$ , and to draw through  $A$  and  $B$  parallels to  $Om_1$  and  $On_1$ , which will meet in  $Q_1$  on the given line  $L$ .

This problem is sometimes of importance in questions of Graphic Statics.

If  $A$  and  $B$  are on opposite sides of the line  $L$ , the conic is an hyperbola if  $O$  is situated in the acute angle formed by the lines  $Sa$  and  $Sb$ ; and conversely if  $A$  and  $B$  are on the same side of  $L$ .

The conic will be a parabola when parallels to  $Sa$ ,  $Sb$ , through  $A$  and  $B$ , meet in the same point on  $L$ .

#### EXAMPLES ON CHAPTER VIII.

1. Given on a conic three points  $A, B, C$  and three other points  $a, b, c$ , determine on the conic a point  $P$  such that  $\{ABCP\} = \{abcP\}$ .

[Either of the points in which the Pascal line (Prob. 85) meets the conic may be taken as the point  $P$ , i.e. draw  $Ac$ ,  $Ca$  intersecting in  $K$ ,  $Bc$ ,  $Cb$  intersecting in  $L$ , and  $KL$  will cut the curve in the required point. For if  $cC$  meets  $KL$  in  $M$ , the anharmonic ratio of  $ABCP$  is that of the pencil  $c(ABCP)$ , i.e. of the range  $KLMP$ , and the anharmonic ratio of  $abcP$  is that of the pencil  $C(abcP)$ , i.e. again of the range  $KLMP$ .]

2. Inscribe in a given conic a triangle with its sides passing through three given points  $ABC$ .

[Draw any line through  $A$  meeting the conic in  $a$  and  $b$ , draw  $bB$  meeting the curve in  $c$ , and draw  $Cc$ , which will not in general pass through  $a$  but will meet the curve in  $d$ . Repeat this twice, giving a range  $aa_1a_2$  and a second  $aa_1a_2$ . Find a point  $P$  such that  $\{aa_1a_2P\} = \{aa_1a_2P\}$ .  $P$  will be a vertex of one such triangle.]



3. Given two straight lines  $AL$ ,  $BL$ , draw a transversal meeting them in  $F$  and  $f$ , so that  $Ff$  shall subtend given angles  $\alpha$  and  $\beta$  at two given points  $P$  and  $p$ .

[From the point  $A$  draw  $AP$ ,  $Ap$  and construct the angles  $APa = \alpha$  and  $Apb = \beta$ , the points  $a$  and  $b$  being on  $BL$ . Imagine  $A$  to slide along  $AL$  and  $a$  and  $b$  will form two homographic divisions, and each of the double points gives a solution.]

4. Given two straight lines  $AL$ ,  $BL$ , draw a transversal meeting them in  $F$  and  $f$  passing through a given point  $P$ , and such that  $Ff$  subtends a given angle  $\beta$  at a second given point  $p$ .

[Last example, the angle  $\alpha$  being zero.]

5. Determine on a given straight line a segment which shall subtend given angles at two given points.

[The two lines of Ex. 3 coincide in direction.]

6. Determine on a given straight line a segment of given length which shall subtend a given angle at a given point.

7. Given two straight lines  $AL$ ,  $BL$  and a point  $P$ , draw through  $P$  transversals cutting  $AL$  in  $F$ ,  $G$  and  $BL$  in  $f$ ,  $g$ , so that  $FG$ ,  $fg$  are given lengths.

[Draw  $AP$  meeting  $BL$  in  $a$ . On  $AL$  make  $AA_1 = FG$ , and on  $BL$  make  $aa_1 = fg$ ; draw  $Pa_1$  cutting  $AL$  in  $a$ ;  $A_1$  and  $a$  will form homographic ranges, the double points of which give solutions.]

8. Given on two straight lines  $AL$ ,  $BL$ , two homographic ranges; draw through two given points  $P$  and  $p$ , lines  $Pa$ ,  $pa_1$  passing through homologous points  $a$ ,  $a_1$  of the ranges and containing between them a given angle  $\theta$ .

[Take any point  $A$  on the first line and its homologous point  $B$  on the second. Draw  $pB$  and  $Pa$  meeting  $AL$  in  $a$  and making an angle  $\theta$  with  $pB$ . Imagine  $A$  to slide along  $AL$  to  $A_1$  and  $A_2$ , giving corresponding positions  $a_1$ ,  $a_2$ , of  $a$ .

The range  $aa_1a_2$  is homographic with  $AA_1A_2$ , because the pencils  $p(BB_1B_2)$  and  $P(aa_1a_2)$  are equiangular.

The double points of these ranges give two solutions.]

9. Given two lines  $Aa$ ,  $Ab$  intersecting in  $A$ , the point  $a$  being fixed; given also a point  $S$  on the opposite side of  $Ab$  from  $a$ . Draw a line through  $S$  meeting  $Ab$  in  $P$  and  $Aa$  in  $p$ , and so that  $AP = ap$ .

[Problem 126, the fixed points being  $A$  and  $a$ , and the ratio one of equality.]

10. Two lines  $OAB$ ,  $Oab$  meet in  $O$ ;  $A$ ,  $B$ ,  $a$  and  $b$  are fixed points on the lines. If  $OAB$  remains fixed and  $Oab$  turns round  $O$ , shew that the locus of the intersection ( $S$ ) of  $Aa$  and  $Bb$  is a circle having its centre ( $C$ ) on  $AB$ , determined by drawing through any one position of  $S$  a parallel  $SC$  to the corresponding position of  $Oab$ .

[The anharmonic ratio of  $A$ ,  $B$ ,  $O$ ,  $C$  is equal to the anharmonic ratio of  $abO$  and an infinitely distant point, so that  $C$  is fixed.]

11. Given two homographic ranges  $ABC... abc...$  on two lines, determine two homologous segments  $KL$ ,  $kl$  which shall subtend given angles  $KPL = \alpha$ ,  $kpl = \beta$  at two given points  $P$ ,  $p$ .

[Take any point, as  $A$ , on the first range and construct the angle  $APF = \alpha$ , and let  $a$  and  $f$  be the homologous points on the second range to  $A$  and  $F$  on the first. Construct the angle  $apf_1 = \beta$ , where  $f_1$  is on the second range. Suppose the point  $A$  to slide along the first range and the points  $f$  and  $f_1$  will form on the second range two homographic divisions, the double points of which will evidently determine two solutions of the question. Three angles, such as  $APF$ , have to be drawn to furnish three pairs of points on the second range.]

12. Given two homographic ranges  $ABC... abc...$  on two lines, determine two homologous segments  $KL$  and  $kl$  of given lengths.

[The principle of the solution of the last example is evidently applicable.]

## CHAPTER IX.

### PLANE SECTIONS OF THE CONE AND CYLINDER.

DEF. If any fixed point  $V$  be taken on a straight line passing through the centre  $O$  of a circle perpendicular to the plane of the circle, and a straight line move so as always to pass through the circumference of the circle and through the point  $V$ , the surface generated by the moving line is called a *Right Circular Cone*, and the line  $OV$  the axis of the cone. If any solid be conceived as divided into any two parts by any plane passing through the solid, the resulting plane surfaces of the solid in contact with the cutting plane are termed *sections* of the solid.

The most convenient way of treating any question on the sections of any solid figure, is by obtaining the projections of the solid on two planes at right angles to each other, the projection of a figure on any plane being, as already explained, the area traced out on the plane by perpendiculars drawn from all points of the figure to the plane.

The projection of any figure on a horizontal plane is called *its plan*, and on a vertical plane an *elevation* of the figure. In any given position therefore a solid can have but one plan but it may have any number of elevations, so that it is always possible to take the vertical plane on which an elevation is projected perpendicular to any plane of section of the solid.

For simplicity, the circular base of the cone will be supposed to be horizontal, and the vertical plane of projection perpendicular to the plane cutting the cone.

In figure 132 (p. 242) let  $o$  be the centre, and  $aob$  a diameter of a circle representing the base of a right circular cone resting on the plane of the paper; draw any line  $xy$  parallel to  $aob$ , and imagine the part of the paper above  $xy$  to be turned up along  $xy$  so as to stand perpendicularly to the part in front of that line; draw  $aa'$ ,  $oo'v'$ , and  $bb'$  all perpendicular to  $xy$  and meeting it in  $a'$ ,  $o'$  and  $b'$  respectively;  $a'$  and  $b'$  will be respectively the elevations of  $a$  and  $b$  the extremities of the circular base,  $o'$  will be the elevation of the centre, and  $o'v'$  will be the elevation of the axis of the cone. The plan of the axis is obviously the point  $o$ . Let the vertex of the cone be at the height  $o'v'$  above the circular base  $ab$ , then  $v'$  will be the elevation of the vertex; and if the lines  $a'v'$ ,  $b'v'$  be drawn and produced indefinitely the triangle  $a'v'b'$  will be the elevation of the portion of the cone between the vertex and the horizontal plane of projection. The angle  $a'v'b'$  is called the vertical angle of the cone, and since the line  $v'o'$  evidently bisects this angle either of the angles  $a'v'o'$ ,  $b'v'o'$  will be the semi-vertical angle.

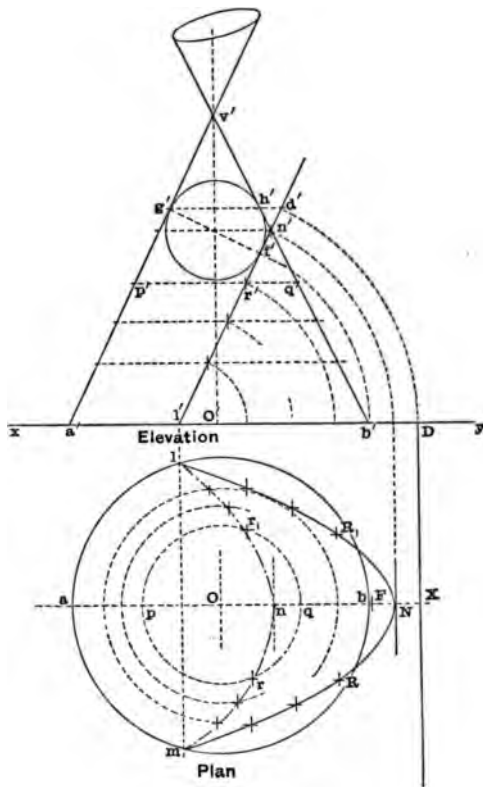
It is evident that any section by a plane perpendicular to the axis, or parallel to the base of the cone, is a circle, the circle becoming infinitely small (i.e. the section being a point) when such plane passes through the vertex; and that the section by any plane through the vertex which cuts the cone in any other point (i.e. which lies within the vertical angle of the cone) will be two straight lines, the angle between which is greatest when the plane passes through the axis, in which case the angle is equal to the vertical angle, and the section is called a Principal Section.

**PROBLEM 130.** *To determine the section of a cone by a plane which does not contain the axis, and does not pass through the vertex.*

*Case I.* Suppose the angle which the plane makes with the horizontal plane to be equal to the angle  $v'a'o'$ , the base angle of the cone, fig. 132. Let the plane intersect the base of the cone in the line  $lm$  perpendicular to  $xy$ , the points  $l$  and  $m$  being on the base of the cone, i.e. on the circle  $ab$ , and let the line  $lm$

meet  $xy$  in  $l'$ ; draw  $l'n'$  parallel to  $a'v'$  meeting  $b'v'$  in  $n'$ . The line  $lm$  is the *horizontal trace*, and  $l'n'$  the *vertical trace* of the

Fig. 132.



section plane, those being the lines in which it cuts the planes of projection respectively. The *plan* of the point  $n'$  will evidently be on the line  $ob$  vertically below  $n'$ , i.e. if  $n'n$  be drawn perpendicular to  $xy$  meeting  $ob$  in  $n$ ,  $n$  will be the plan of  $n'$ .

Imagine a horizontal plane to cut the solid at any height between the base and the point  $n$ , as at  $p'q'$ ; it will evidently cut the cone in a circle of diameter  $p'q'$ , and which would in

plan have  $o$  for its centre, and it will cut the section plane in a horizontal line the elevation of which is the point  $r'$ , and the plan of which is the line  $rr_1r'$  perpendicular to  $xy$ . The points of intersection of the circle and line will evidently be points on the desired curve of intersection, and therefore if  $r, r_1$  are the points in which the plan of the line cuts the circle  $pq$  described with centre  $o$  and radius equal to  $\frac{1}{2}p'q'$ , these will be points on the projected curve of intersection. Similarly any number of additional points can be found by taking a series of planes parallel to  $p'q'$ . The curve  $lr_1nrm$  will be of course the plan of the required curve.

Now imagine the plane of section to be rotated round its horizontal trace  $lm$  until it coincides with the horizontal plane of projection, carrying with it the various points of intersection as found. In elevation they would of course travel over circular arcs described with  $l'$  as centre and with radii equal to the distances between  $l'$  and their respective elevations, while on plan they would travel along lines through their respective plans perpendicular to  $lm$ .

The point  $n$  would therefore reach  $N$ , the points  $r$  and  $r_1$  would reach  $R$  and  $R_1$  and so on, and the curve  $lR_1NRm$  would be the *true form* of the section made by the given plane.

Inscribe in the cone a sphere which will also touch the given plane of section. The *elevation* of such sphere will be the circle touching  $v'a', v'b'$  and  $n'l'$ ; let it do so in the points  $g', h'$  and  $f'$ , and let the line  $g'h'$  meet  $l'n'$  in  $d'$ .

$d'$  will be the elevation of the line of intersection of the given section plane and of the plane through the circle of contact of the cone and the inscribed sphere. On being turned down along with the plane of section this line would therefore come into the position  $DX$ , while the point  $f'$  would come to  $F$ .

The required curve of intersection is a parabola having  $F$  for focus and  $DX$  for directrix.

*Proof.* The line whose elevation is  $f'r'$ , is a tangent to the inscribed sphere, since it lies in a tangent plane to that sphere (the given section plane) and passes through the point of contact



of that plane. It is therefore equal in length to any other tangent to the sphere drawn from the point whose elevation is  $r'$ , and since  $r'$  is really on the surface of the cone, the length of the tangents drawn from it to the sphere must be equal to the line  $h'q'$ , which is evidently equal to  $d'r'$ , i.e.  $FR = r'd' =$  the perpendicular distance of  $R$  from  $DX$ ; therefore  $R$  is a point on the parabola described with focus  $F$  and directrix  $DX$ , and the same of course holds for any other point of the curve.

*Case II.* Let the angle which the plane of section makes with the horizontal plane be less than the angle  $v'a'o'$ , the base angle of the cone (fig. 133).

Proceeding exactly as before, let the plane intersect the plane of the base of the cone in the line  $lm$  perpendicular to  $xy$ , and draw  $ln'n''$  making any angle less than the base angle of the cone with  $xy$  and meeting  $v'a'$ ,  $v'b'$  in  $n'$  and  $n''$ .

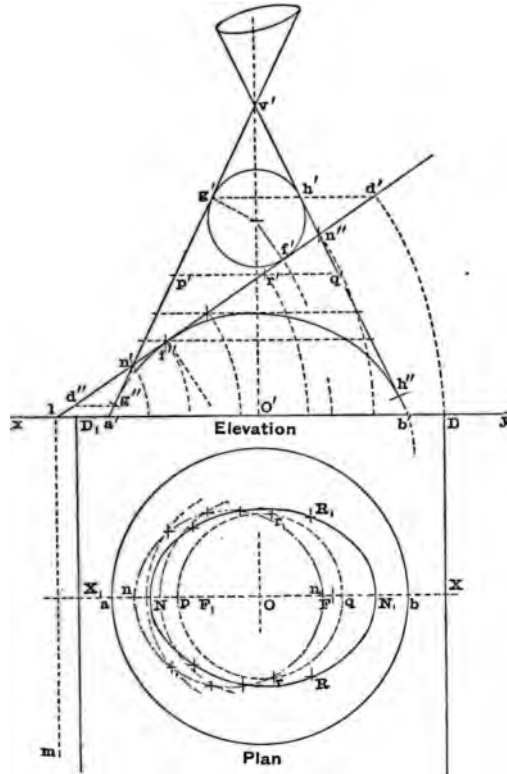
Take any horizontal plane (as  $p'q'$ ) at any height between  $n'$  and  $n''$ , meeting  $v'a'$  in  $p'$ ,  $v'b'$  in  $q'$ , and  $ln'$  in  $r'$ , and draw the plan of the circle in which this plane cuts the cone (the circle described with centre  $o$  and radius  $op$ , or  $oq = \frac{1}{2}p'q'$ ) and of the line in which it cuts the section plane (through  $r'$  perpendicular to  $xy$ ). The points of intersection  $r$  and  $r_1$  of this circle and line will be the *plans* of two points on the required curve of section. Turning the plane round its horizontal trace until it coincides with the horizontal plane the point  $n'$  reaches  $N$ , the point  $n''$  reaches  $N_1$ , and  $r$  and  $r_1$  come to  $R$  and  $R_1$  respectively. Similarly any number of points can be found, and it will be found that they lie on a closed curve.

Inscribe in the given cone spheres to touch also the given plane of section (two such can be drawn, one above and the other below the plane); let them touch the plane in  $f'$  and  $f''$ , and  $v'a'$ ,  $v'b'$  in  $g'$ ,  $g''$ , and  $h'$ ,  $h''$  respectively: let  $g'h'$  meet  $ln'$  in  $d'$ , and  $g'h''$  meet it in  $d''$ .

$d'$  and  $d''$  will be the elevations of the lines of intersection of the given plane of section with the planes of contact of the cone and its inscribed spheres.

Suppose  $f'$  and  $f''$  and the lines through  $d'$  and  $d''$  to be turned down along with the plane of section so that  $f'$  comes to  $F$ ,  $f''$  to  $F_1$ , and the lines to  $DX$  and  $D_1X_1$ , respectively, and the curve of section will be an ellipse with foci  $F$  and  $F_1$  with directrices  $DX$  and  $D_1X_1$ .

Fig. 133.



*Proof.* The line whose elevation is  $f'r'$  is a tangent to the inscribed sphere, since it lies in a tangent plane to that sphere (the given section plane) and passes through the point of contact of that plane. It is therefore equal in length to any other



tangent to the sphere drawn from the point whose elevation is  $r'$ ; and since  $r'$  is really on the surface of the cone, the length of the tangents drawn from it to the sphere must be equal to the line  $h'q'$ , which is always in a constant ratio to but is less than  $d'r'$ , since the angle  $d'r'q'$  is less than  $h'q'r'$ , i. e.  $FR$  is always in a constant ratio, smaller than unity, to the perpendicular distance of  $R$  from  $DX$ .

Therefore  $R$  is a point on the ellipse described with focus  $F$  and directrix  $DX$ , and the same of course holds for any other point on the curve.

*Case III.* Let the angle which the plane of section makes with the horizontal plane be greater than the angle  $v'a'o'$ , the base angle of the cone (fig. 134).

The description of the last case applies exactly to the present, and the figure is lettered to correspond. The plane will necessarily cut both sheets of the cone, and the curve will consist of two infinite branches.

It will be found to be an hyperbola with foci  $F$  and  $F_1$  and with directrices  $DX$  and  $DX_1$ .

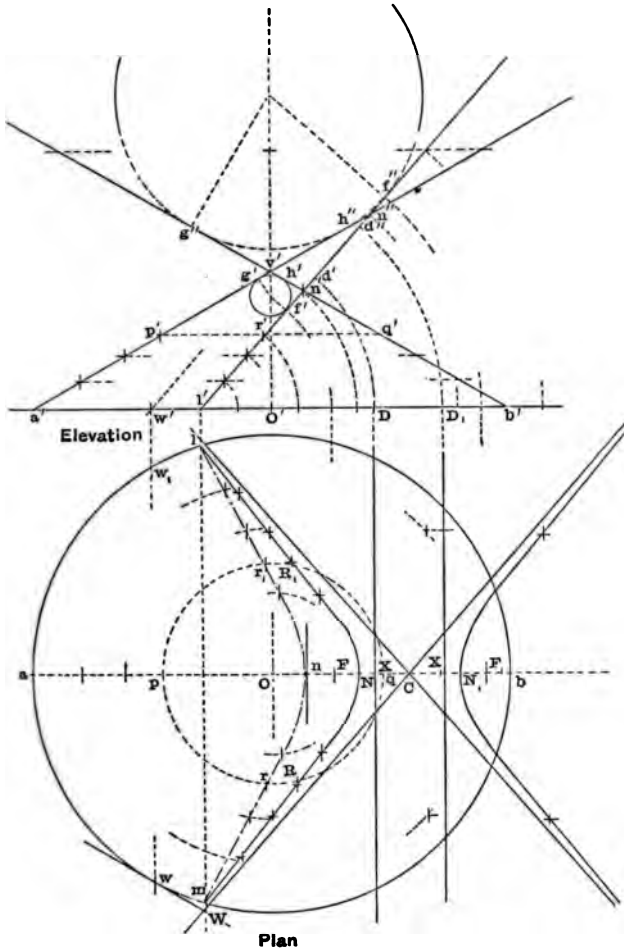
*Proof.* In this case the line  $h'q'$  is always in a constant ratio to but is greater than the line  $d'r'$ . Hence the distance of any point on the curve from the focus  $F$  is always in a constant ratio, greater than unity, to its distance from the directrix  $DX$ .

The two straight lines in which a cone is intersected by a plane through the vertex parallel to an hyperbolic section are parallel to the asymptotes of the hyperbola.

The asymptotes may be thus found. They of course pass through the point  $C$  midway between  $X$  and  $X_1$ . Draw the generators of the cone parallel to the given section plane, i. e. draw  $v'w'$  parallel to  $l'n'$ , which will be the elevation of such generators, and project  $w'$  to  $w$  and  $w_1$  on the base of the cone. The tangent planes to the cone along the generators whose plans are  $ow$ ,  $ow_1$  and whose elevations are  $v'w'$  will intersect the given plane of section in the required asymptotes. If therefore a tangent at  $w$  to the circular base of the cone meet  $lm$  (the hori-

zontal trace of the given section plane) in  $W$ ,  $W$  will be a point on one asymptote, which will therefore be the line  $CW$ . Similarly the second asymptote can be obtained from the point  $w_1$ .

Fig. 134.





but  $f$  is the focus,  $q$  the vertex, and  $x$  the trace of the directrix of the section made by the plane  $geg$ .

If the conic is to be an hyperbola, i. e. if  $m > n$ , there is a limit to the vertical angle of the cone in order that the problem may be possible. It will be observed that the length  $eg$  is  $\frac{n}{\cos a}$ , where  $a$  is the semi-vertical angle of the cone, and  $eg$  must evidently be greater than  $eo'$  or  $m$ .

Therefore  $\frac{n}{m}$  must be greater than  $\cos a$  or  $a > \cos^{-1} \frac{n}{m}$ , i. e.  $a$  must be greater than the angle whose cosine is  $\frac{n}{m}$ , or in other words the ratio of the height of the cone to length of slant side must be less than  $\frac{n}{m}$ .

**PROBLEM 132.** *From a given cone to cut a conic of given eccentricity and having a given distance  $FX$  between focus and directrix (Fig. 135).*

As in the last problem, draw some one plane of section of the required eccentricity, as  $geg$ , and determine its focus  $f$  and the trace ( $x$ ) of its directrix.

Draw  $v'x$ ,  $v'f$  to the vertex of the cone; on  $xf$  make  $xf_1 =$  the given distance  $FX$ , and through  $f_1$  draw  $f_1F$  parallel to  $xv'$  meeting  $fv'$  in  $F$ . Through  $F$  draw a line parallel to  $gq$  meeting the slant sides of the cone in  $A$  and  $A_1$  and  $xv'$  in  $X$ . This will be the trace of the required plane of section,  $A$  and  $A_1$  being the vertices,  $F$  a focus, and  $X$  the trace of one of the directrices.

**DEF.** If a straight line move so as to pass through the circumference of a given circle, and to be perpendicular to the plane of the circle, it traces out a surface called a *Right Circular Cylinder*. The straight line drawn through the centre of the circle perpendicular to its plane is the *axis* of the cylinder.

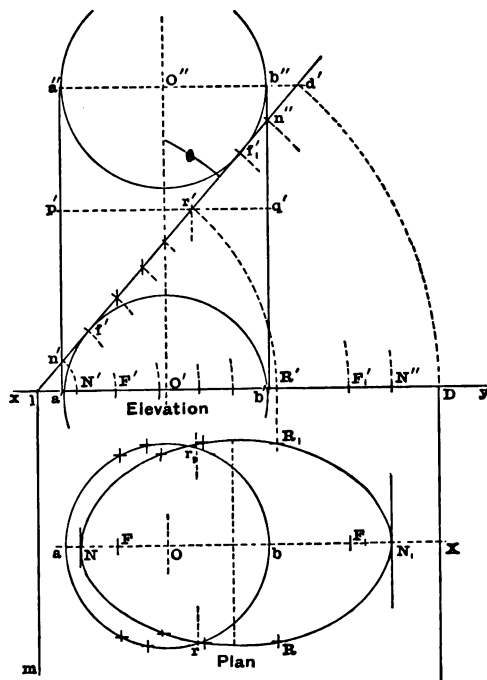
The cylinder may evidently be regarded as a particular case of the cone, the vertex being at an infinite distance from the base so that the generators are ultimately parallel.

As with the right circular cone, it is evident that a section of the surface by any plane perpendicular to the axis is a circle, and that a section by any plane parallel to the axis (i.e. passing through the infinitely distant vertex) consists of two parallel lines.

**PROBLEM 133.** *To determine the section of a right circular cylinder by a plane inclined at any given angle ( $\theta$ ) to the axis (Fig. 136).*

Let  $lm$  be the line of intersection of the given plane of section with the horizontal plane of the base of the cylinder, i.e. the

Fig. 136.



horizontal trace of the plane of section. Draw any ground line  $xy$  perpendicular to  $lm$ , and through  $l$  draw  $ld'$  making the angle

$d'l$  equal to the complement of the given angle  $\theta$ . Let  $o$  be the plan of the axis of the cone, and through  $o$  draw  $oo''$  perpendicular to  $xy$ ;  $o'o''$  will be the elevation of the axis of the cone on the vertical plane of projection, and  $ld'$  will be the trace on the same plane of the given section plane.

With centre  $o$  and radius equal to that of the cylinder describe a circle  $ab$ , and draw  $aob$  perpendicular to  $lm$ ; through  $a$  and  $b$  draw  $aa''$ ,  $bb''$  perpendicular to  $xy$ , meeting it in  $a'$  and  $b'$ , and the rectangle  $a'a'b'b''$  will be the elevation of the cylinder. Let  $ld'$  cut  $a'a''$  in  $n'$  and  $b'b''$  in  $n''$ .

Imagine a horizontal plane to cut the solid at any height between  $n'$  and  $n''$ , as at  $p'q'$ ; it will evidently cut the cylinder in a circle of diameter  $p'q'$ , and which would in plan have  $o$  for its centre, and it will cut the section plane in a horizontal line the elevation of which is the point  $r'$ , in which  $p'q'$  cuts  $ld'$  and the plan of which is the line  $rr_1r'$  perpendicular to  $xy$ . The points of intersection of the circle and line will evidently be points on the desired curve of intersection, and therefore if  $r, r_1$  are the points in which the plan of the line cuts the circle  $ab$  (which is of course the *plan* of the circle  $p'q'$ ) these will be the *plans* of the points in which the horizontal plane at the height  $a'p'$  above the base of the cylinder cuts the required curve of intersection.

Now imagine the plane of section to be rotated round its horizontal trace  $lm$  until it coincides with the horizontal plane of projection. In elevation the point  $r'$  would travel over the circular arc  $r'R'$  struck with  $l$  as centre, meeting the ground line in  $R'$ , while on plan the points  $r$  and  $r_1$  would travel along lines  $rR, r_1R$ , perpendicular to  $lm$  reaching the horizontal plane of projection in the points  $R, R_1$  found by drawing  $R'R_1R$  perpendicular to  $xy$ .

Similarly any number of additional points can be found by drawing a series of planes parallel to  $p'q'$ , all of which will of course cut the cylinder in circles, the plans of which are the circle  $ab$ .



The point  $n'$  travels in elevation over the arc  $n'N'$ , and the plan of  $N'$  is simultaneously on  $ab$  and on  $N'N$  perpendicular to  $xy$ ; and the point  $n''$ , the plan of which is  $b$ , similarly reaches the horizontal plane at  $N_1$ .

The required curve of intersection is an ellipse having  $NN_1$  for major axis, and for minor axis a length equal to the diameter of the cylinder.

The minor axes of all ellipses which can be cut from the same cylinder are consequently of equal length, but the length of the major axis depends jointly on the diameter of the cylinder and the inclination of the cutting plane to its axis, since

$$n'n'' = a'b' \operatorname{cosec} \theta.$$

Just as in the case of the cone, if spheres be inscribed in the cylinder touching the plane of section they will do so in the foci of the curve of intersection. The elevations of these spheres are the circles shewn in the figure touching the line  $ld'$  in the points  $f''$  and  $f_1'$ , and also touching  $a'a'', b'b''$ .  $f''$  travels over the circular arc  $f''F''$ , and  $F''F''$  perpendicular to  $xy$ , meeting  $ab$  in  $F$ , determines  $F$ , one of the foci.

The horizontal planes through the circles of contact of the spheres and cylinder intersect the plane of section in the directrices of the curve.  $d'$  is therefore the elevation of one of them, which after rotation of the section plane round  $lm$  comes into the position  $DX$ .

*Proof.* The line whose elevation is  $f_1'r'$  is a tangent to the inscribed sphere, since it lies in a tangent plane to that sphere (the given section plane) and passes through the point  $f_1'$  in which the sphere touches the plane. It is therefore equal in length to any other tangent to the sphere drawn from the point whose elevation is  $r'$ , and since  $r'$  is really on the surface of the cylinder, the length of the tangents drawn from it to the sphere must be  $r'k'$ , where  $r'k'$  is parallel to the axis of the cylinder, and  $k'$  is on the circle of contact of sphere and cylinder. But  $r'k' : r'd'$  in a constant ratio  $= \cos \theta$ , and  $r'k' = F_1R$ ;  $r'd' = RM$ , where  $RM$  is perpendicular to  $DX$  meeting it in  $M$ , therefore  $F_1R : RM$  in a constant ratio, or the locus of  $R$  is an ellipse.

## THE OBLIQUE CYLINDER.

**DEF.** If a straight line, which is not perpendicular to the plane of a given circle, move parallel to itself, and always pass through the circumference of the circle, the surface generated is called an oblique cylinder.

The line through the centre of the circular base parallel to the generating lines is the axis of the cylinder.

The section of the cylinder made by a plane containing the axis and perpendicular to the base is called the *principal section*.

The section of the cylinder by a plane perpendicular to the principal section, and inclined to the axis at the same angle as the base, is called a *sub-contrary section*.

It is evident that any section by a plane parallel to the axis consists of two parallel lines, and that any section by a plane parallel to the base is a circle.

**PROBLEM 134.** *To determine the sub-contrary section of an oblique cylinder.*

Let  $o$  (fig. 137) be the centre of the circular base, and the circle on  $ab$  as diameter the base of the cylinder; let  $ob$  be the plan of the axis. Draw  $xy$  parallel to  $ab$ , so that the elevation on  $xy$  as ground line will be parallel to the principal section of the cylinder; draw  $aa'$ ,  $oo'$ ,  $bb'$  perpendicular to  $xy$  meeting it in  $a'$ ,  $o'$ ,  $b'$ , which will be the elevations of the corresponding points of the base. Since the elevation is parallel to the principal section, the angle which the elevation of the axis (i.e. the line  $o'c'$ ) makes with the ground line will be the real angle which the axis itself makes with the horizontal plane. Draw  $a'a_1'$ ,  $b'b_1'$  parallel to  $o'c'$ , these lines are the elevations of the bounding lines of the solid projected on the vertical plane standing on  $xy$ . Draw any line  $a_1'b_1'l$  making the same angle  $\theta$  with  $o'c'$  as  $o'c'$  makes with  $xy$ , meeting  $xy$  in  $l$ , and draw  $lm$  perpendicular to  $xy$ .

$lm$  will be the horizontal trace and  $la_1'$  the vertical trace of a plane of sub-contrary section; and if this plane be rotated round  $lm$  till it coincides with the horizontal plane, every point on the

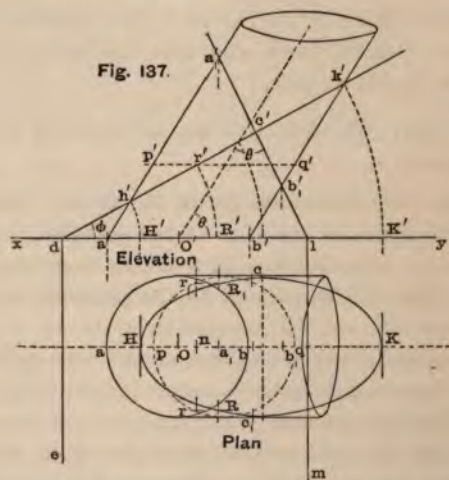


surface of the cylinder between  $a_1'$  and  $b_1'$  will evidently reach a point on the circle on  $ab$  as diameter, i.e. the true form of the sub-contrary section is a circle.

The horizontal projection of the sub-contrary section is the ellipse having  $cc_1$  projected from  $c'$ , the point in which  $a_1'b_1'$  intersects the axis of the cylinder as major axis, and  $a_1b_1$  the projection of  $a_1'b_1'$  as minor.

**PROBLEM 135.** *To determine the section of an oblique cylinder by a plane not parallel to the axis, to the base, or to a sub-contrary section.*

*Case I.* Let the plane of section be perpendicular to the principal section (fig. 137).



The horizontal trace ( $de$ ) of the plane of section must be drawn perpendicular to  $ab$ , the plan of the axis. If the plane of section makes an angle ( $\phi$ ) with the horizontal plane, the vertical trace must be drawn through  $d$  making this angle with  $xy$ . Let it meet  $a'a_1'$  in  $h'$  and  $b'b_1'$  in  $k'$ .

Draw any circular section, as  $p'q'$ , between  $h'$  and  $k'$  meeting  $dk'$  in  $r_1'$ ; the plan is of course the circle on  $p'q'$  as diameter pro-

jected from  $p'$  and  $q'$  on  $ab$ , and if the projection of the point  $r'$  cuts this circle in  $r$  and  $r_1$ , these will be the *plans* of two points on the required curve. If  $rr_1$  meet  $pq$  in  $n$  we have  $rn^2 = pn \cdot nq$ . If now the plane of section be rotated round  $de$  till it coincides with the horizontal plane,  $h'$  travels in elevation to  $H'$  and in plan to  $H$ ,  $k'$  travels in elevation to  $K'$  and in plan to  $K$ , and  $r$  and  $r_1$  reach  $R$  and  $R_1$  respectively. Therefore if  $RR_1$  meet  $ab$  in  $N$ ,

$$RN^2 = pn \cdot nq = p'r' \cdot r'q';$$

but  $p'r' : h'r'$  in a constant ratio,

and  $r'q' : r'k'$  in the same ratio,

$\therefore p'r' \cdot r'q' : h'r' \cdot r'k'$  in a constant ratio;

but  $h'r' \cdot r'k' = HN \cdot NK$ ,

$\therefore RN^2 : HN \cdot NK$  in a constant ratio,

or the locus of  $R$  is an ellipse (Prop. 4, p. 108).

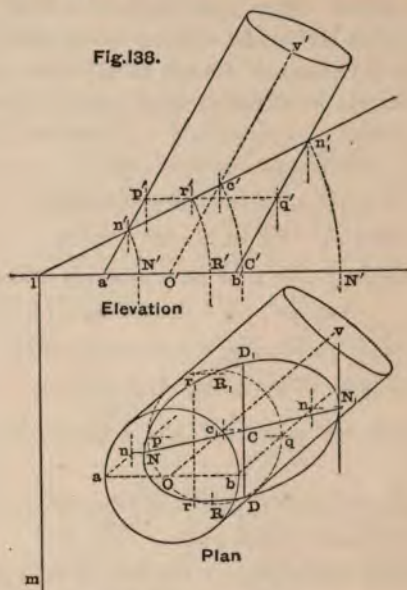
*Case II.* Let the plane cut the cylinder in any manner (fig. 138).

Let  $ab$  be the diameter of the base perpendicular to the horizontal trace of proposed section plane.

The circle on  $ab$  is the plan of the base of the cylinder,  $ov$  the plan, and  $o'v'$  the elevation of its axis, the elevation being projected on a plane perpendicular to the proposed section plane. Lines through  $a$  and  $b$  parallel to  $ov$  are of course the plans of the generators through  $a$  and  $b$ , and if  $a$  and  $b$  are projected on to the ground line at  $a'$  and  $b'$  lines through these points parallel to  $o'v'$  will be the elevations of these generators and will be the bounding lines of the solid as seen in the proposed elevation.

$lm$  is the horizontal trace, and  $ln_1'$  the vertical trace of the section plane; let  $a'n'$  parallel to  $o'v'$  meet  $ln_1'$  in  $n'$ , and  $b'n_1'$  meet it in  $n_1'$ ;  $n'$  and  $n_1'$  are evidently points on the required curve of intersection, and their plans  $n$  and  $n_1$  are found by projecting  $n'$  and  $n_1'$  on to the plans of the generators through  $a$  and  $b$ . Take any horizontal section of the cylinder between  $n'$  and  $n_1'$ , as  $p'q'$ ; the plan is of course a circle of diameter  $p'q'$ , and its position can be determined by projecting  $p'$  and  $q'$  on to the plans

of the generators through  $a$  and  $b$ , as at  $p, q$ . This horizontal section and the proposed section plane intersect in a line the



elevation of which is  $r'$ , the point in which  $p'q'$  cuts  $ln'_i$ , and the plan of this line cuts the circle  $pq$  in points  $r$  and  $r_1$  projected from  $r'$ , which are *plans* of points on the required curve of intersection.

Now rotate the section plane round  $lm$ , its horizontal trace, till it coincides with the horizontal plane: in elevation the points  $n', r', n'_i$  travel over circular arcs to  $N', R', N'_i$ ; in plan  $n, r, r_1, n_1$  travel over lines perpendicular to  $lm$  to  $N, R, R_1, N_1$ , obtained by projecting  $N', R'$  and  $N'_i$ .

These are points situated on the true outline of the curve of intersection, and any additional number of points can be obtained in precisely the same manner. The curve is an ellipse having  $NN_i$  as a diameter and  $RR_1$  as a corresponding double ordinate, so that  $DD_1$ , the diameter conjugate to  $NN_i$ , can at once be drawn

by bisecting  $NN_1$  in  $C$ , drawing through  $C$  a parallel to  $RR_1$  or to  $lm$ , and making on it  $CD = CD_1 = ao$  the radius of circular base of cylinder. That the curve is an ellipse may be proved similarly to Case I.

#### THE OBLIQUE CONE.

DEF. If a straight line pass always through a fixed point and the circumference of a fixed circle, and if the fixed point be not in the straight line through the centre of the circle at right angles to its plane, the surface generated is called an *oblique cone*.

The fixed point is called the *vertex* and the line joining the vertex to the centre of the circle the *axis* of the cone.

The section of the cone made by a plane containing this axis and perpendicular to the circular base, is called the *principal section*.

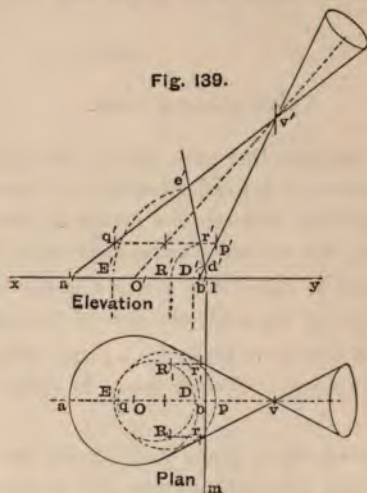
The section made by a plane not parallel to the base, but perpendicular to the principal section, and inclined to the generating lines in that section at the same angles as the base, is called a *sub-contrary section*.

PROBLEM 136. To determine the sub-contrary section of an oblique cone (Fig. 139).

Let  $o$  be the centre and  $oa$  the radius of the circular base, and let  $ov$  be the plan of the axis. Draw a ground line  $xy$  parallel to  $ov$ , and let  $v'$  be the elevation of the vertex on a vertical plane standing on  $xy$ . Project  $o$  to  $o'$ , and the circular base to  $a'b'$ , so that  $o'v'$  will be the elevation of the axis, and  $a'v'b'$  the outline of the cone;  $a'v'b'$  is evidently also identical with the principal section.

Draw any line  $e'd'l$  making the angle  $a'e'l =$  the angle  $v'b'y$ , and meeting  $v'b'$  in  $d'$  and  $xy$  in  $l$ ; the angle  $e'd'v'$  is evidently equal to the angle  $v'a'b'$ , so that  $e'l$  may be taken as the vertical trace of the plane of a sub-contrary section, the horizontal trace of which must be the line  $lm$  perpendicular to  $xy$ .

Take any horizontal section as  $p'q'$  between  $d'$  and  $e'$ , the plan of which will be a circle on  $pq$  as diameter,  $p$  and  $q$  being



the projections of  $p'$  and  $q'$  on the plan of the axis or central plane of the cone. The plane of this section intersects the plane of sub-contrary section in a straight line, the elevation of which is the point  $r'$  in which  $p'q'$  intersects  $le'$ , and the plan of which is  $rr_1$  projected from  $r'$ . If  $rr_1$  meet the circle on  $pq$  in  $r$  and  $r_1$ , these will be plans of two points on the required curve of sub-contrary section, and if  $rr_1$  meet  $pq$  in  $n$ ,

$$rn^2 = np \cdot nq = r'q' \cdot r'p' = r'd' \cdot r'e';$$

since a circle can be described round  $e'q'd'p'$ .

Rotate the plane of section round its horizontal trace till it coincides with the horizontal plane of projection, and  $e'$ ,  $r'$  and  $d'$  travel to  $E'$ ,  $R'$  and  $D'$  the corresponding positions in plan being  $E$ ,  $R$  and  $R_1$  and  $D$  their projections. These are of course points on the real outline of the required curve, and if  $RR_1$  meet  $ED$  in  $N$ , since

$$RN = rn, \quad EN = e'r', \quad ND = r'd',$$

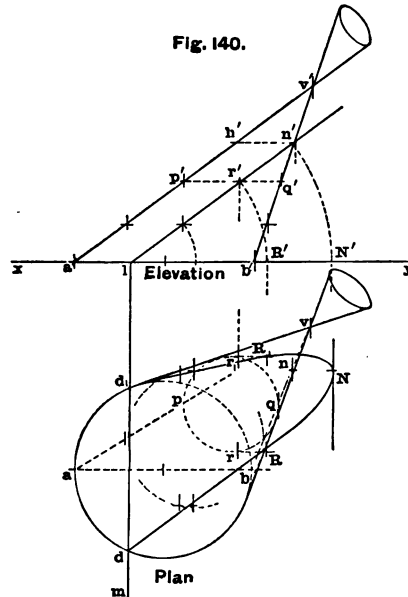
we have  $RN^2 = EN \cdot ND$ ,  
 or the locus of  $R$  is a circle on  $ED$  as diameter,  
 i.e. the sub-contrary section of an oblique cone is a circle.

It is evident that all sections parallel to the base or to the plane  $e'lm$  are also circles.

Planes parallel to the base, or to a sub-contrary section, are called also *Cyclic Planes*.

**PROBLEM 137.** *To determine the section of an oblique cone by a plane not parallel to a cyclic plane and not passing through the vertex (Fig. 140).*

*Case I.* Let the plane be parallel to a tangent plane of the



cone, i.e. let it be parallel to a generator and perpendicular to the plane containing that generator and the axis.



Let  $a'v'b'$  be the elevation of the cone,  $v$  the plan of the vertex the elevation of which is  $v'$ , and  $ab$  the diameter of the circular base parallel to the plane of the elevation.

It is convenient to take the plane of section perpendicular to the plane of the elevation; so that its horizontal trace  $lm$  may be drawn perpendicular to  $xy$ , and its vertical trace must then be drawn parallel either to  $a'v'$  or to  $b'v'$ , since the plane itself must be parallel to one or other of these generators—let  $lm'$  parallel to  $a'v'$  be its vertical trace. If  $lm$  cuts the circle on  $ab$  as diameter in  $d$  and  $d_1$ , these will be points on the required curve of intersection, and if  $lm'$  meets  $b'v'$  in  $n'$ ,  $n'$  will be the elevation of another point, the plan of which will be  $n$ , the intersection of  $bv$  and the projection of  $n'$ .

Draw any horizontal plane as  $p'q'$  between  $l$  and  $n'$ , meeting  $a'v'$  in  $p'$ ,  $b'v'$  in  $q'$  and  $lm'$  in  $r'$ ; this plane cuts the cone in a circle the elevation of which is  $p'q'$ , and the plan of which is a circle on  $pq$  as diameter obtained by projecting  $p'$  and  $q'$  on  $av$  and  $bv$  respectively. It meets the section plane in a line the elevation of which is the point  $r'$ , and the plan of which is the line  $rr_1$  projected from  $r'$ ; if this line meets the circle on  $pq$  in  $r$  and  $r_1$ , these are the *plans* of two points on the required curve of intersection and similarly the plans of any additional number of points can be obtained.

Rotate the section plane round its horizontal trace till it coincides with the horizontal plane of projection; the point  $n'$  travels in elevation to  $N'$  and the point  $r'$  to  $R'$ ; in plan  $n$ ,  $r$ , and  $r_1$  travel along  $nN$ ,  $rR$  and  $r_1R_1$  perpendicular to  $lm$  till they meet the projections of  $N'$  and  $R'$  respectively, and  $d$ ,  $R$ ,  $N$ ,  $R_1$  and  $d_1$  will be points on the real outline of the required curve of intersection. It is a parabola having the tangent at  $N$  parallel to  $RR_1$ .

*Proof.* If  $K$  bisects  $RR_1$ ,  $KR^2 = p'r' \cdot r'q'$ .

Through  $n'$  draw  $h'n'$  parallel to  $p'q'$  meeting  $a'v'$  in  $h'$ , then  $h'n' = p'r'$ ,

$$r'q' : r'n' :: h'n' : h'v';$$

$$\therefore p'r' \cdot r'q' : h'n' \cdot r'n' :: h'n' : h'v',$$

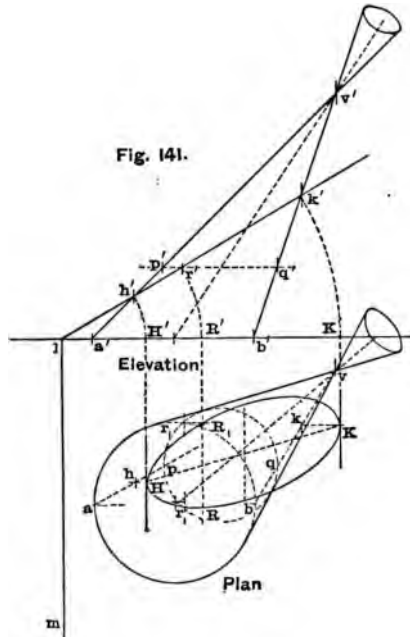
$$\therefore KR^2 : h'n' \cdot r'n' \text{ in a constant ratio,}$$

but  $r'n' = KN \cos \theta$ , where  $\theta$  is the angle between  $KN$  and  $pq$ , and is constant;

$\therefore KR^2 = KN$  multiplied by some constant, or the locus of  $K$  is a parabola.

*Case II.* Let the plane of section meet all the generating lines on the same side of the vertex (Fig. 141).

Let  $a'v'b'$  be the elevation of the cone,  $v$  the plan of  $v'$  the vertex, and  $ab$  the diameter of the circular base parallel to the



ground line and therefore the plan of  $a'b'$ . Let the plane of section be perpendicular to the vertical plane of projection, and



draw its horizontal trace  $lm$  perpendicular to  $xy$  and its vertical trace cutting  $a'v'$  in  $h'$ , and  $b'v'$  in  $k'$ . Project  $h'$  to  $h$  on  $av$  and  $k'$  to  $k$  on  $bv$ , then  $h$  and  $k$  are the *plans* of the points in which the generators through  $a$  and  $b$  meet the section plane, i.e. are the plans of two points on the required curve of intersection.

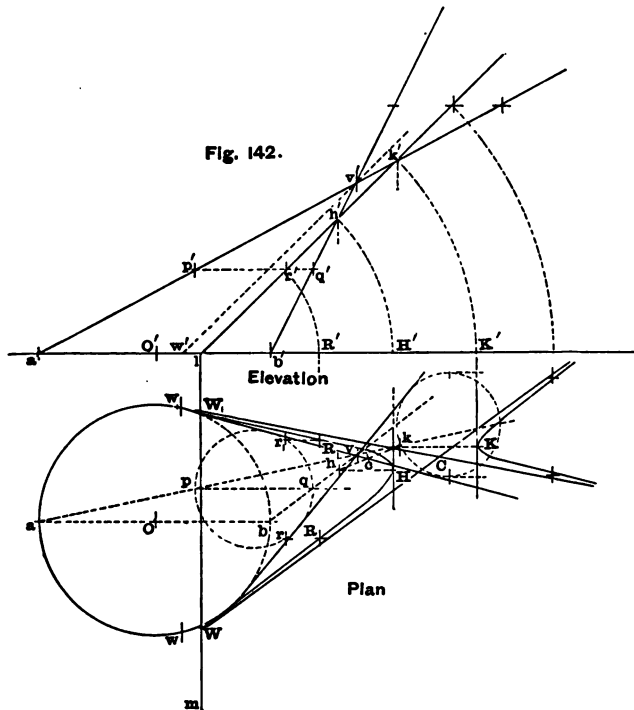
Imagine the cone cut by any horizontal plane as  $p'q'$  between  $h'$  and  $k'$ , the elevation of the curve of intersection will be the line  $p'q'$ , meeting  $a'v'$  in  $p'$  and  $b'v'$  in  $q'$  and  $lk'$  in  $r'$ ; and the plan will be the circle on  $pq$  as diameter, obtained by projecting  $p'$  on  $av$  and  $q'$  on  $bv$ . The required plane of section cuts this plane of circular section in a line the elevation of which is  $r'$ , and the plan of which is  $rr_1$  projected from  $r'$ . If  $rr_1$  meets the circle on  $pq$  in the points  $r$  and  $r_1$ , these are the plans of two points of the required curve of intersection. Similarly the plans of any additional number of points can be obtained.

Rotate the plane of section round its horizontal trace till it coincides with the horizontal plane of projection; in elevation  $h'$ ,  $r'$  and  $k'$  travel to  $H'$ ,  $R'$ , and  $K'$ , and on plan  $h$ ,  $r$ ,  $r_1$  and  $k$  travel along  $hH$ ,  $rR$ ,  $r_1R_1$ ,  $kK$ , perpendicular to  $lm$  till they meet the projections of  $H'$ ,  $R'$  and  $K'$ . The points  $H$ ,  $R$ ,  $K$ ,  $R_1$  are points on the real outline of the required curve of intersection. It is an ellipse having  $HK$  as a diameter, and  $RR_1$  as corresponding double ordinate.

*Case III.* Let the section plane cut both sheets of the cone (Fig. 142).

Let  $a'v'b'$  be the elevation of the cone,  $v$  the plan of  $v'$  the vertex, and  $ab$  the diameter of the circular base parallel to the ground line, and therefore the plan of  $a'b'$ . Let the plane of section be perpendicular to the vertical plane of projection, and draw its horizontal trace  $lm$  perpendicular to  $xy$ , and its vertical trace  $lk'$  cutting  $b'v'$  in  $h'$ , and  $a'v'$  in  $k'$ . Project  $h'$  to  $h$  on  $bv$ , and  $k'$  to  $k$  on  $av$ , then  $h$  and  $k$  are the *plans* of the points in which the generators through  $a$  and  $b$  meet the section plane, i.e. are the plans of two points on the required curve of intersec-

**tion.** Imagine the cone cut by any horizontal plane as  $p'q'$ ; the elevation of the circle in which this plane meets the cone will be



the line  $p'q'$  meeting  $a'v'$  in  $p'$ ,  $b'v'$  in  $q'$ , and  $lk'$  in  $r'$ , and the plan will be the circle on  $pq$  as diameter obtained by projecting  $p'$  and  $q'$  on  $av$  and  $bv$  respectively. The required plane of section cuts this plane of circular section in a line, the elevation of which is  $r'$ , and the plan of which is  $rr_1$  projected from  $r'$ . If  $rr_1$  meets the circle on  $pq$  in the points  $r$  and  $r_1$ , these are the plans of two points of the required curve of intersection. Similarly the plans of any additional number of points can be obtained.

**Rotate the plane of section round its horizontal trace  $lm$  till**

it coincides with the horizontal plane of projection; in elevation  $h'$ ,  $r'$  and  $k'$  travel to  $H'$ ,  $R'$  and  $K'$ , and on plan  $h$ ,  $r$ ,  $r_1$  and  $k$  travel along  $hH$ ,  $rR$ ,  $r_1R_1$ ,  $kK$  perpendicular to  $lm$  till they meet the projections of  $H'$ ,  $R'$  and  $K'$ . The points  $H$ ,  $R$ ,  $R_1$  and  $K$  are points on the real outline of the required curve of intersection. It is an hyperbola having  $HK$  as a diameter, and  $RR_1$  as corresponding double ordinate of the branch through  $H$ .

The asymptotes are parallel to the generators of the cone which are parallel to the plane of section. If therefore  $v'w'$  be drawn parallel to  $lk'$  meeting  $xy$  in  $w'$ , and  $w'$  be projected to meet the circular base  $ab$  in  $w$  and  $w_1$ , the *plans* of the asymptotes will be parallel to  $vw$ ,  $vw_1$ . Bisect  $hk$  in  $c$ , and draw  $cW$ ,  $cW_1$  parallel respectively to  $vw$  and  $vw_1$ , and meeting  $lm$  in  $W$ ,  $W_1$  which will be points on the asymptotes, and they can therefore be drawn through  $C$  the point of bisection of  $HK$ .

#### EXAMPLES ON CHAPTER IX.

1.  $AVA_1$ , an isosceles triangle, obtuse angled at  $V$ , is the elevation of a cone. Shew that if  $VB$  be drawn meeting  $AA_1$  in  $B$ , and such that  $\overline{VB}^2 = AB \cdot BA_1$  (Ex. 15, Chap. II.) and any plane be drawn having its vertical trace parallel to  $VB$ , and horizontal trace perpendicular to  $AA_1$ , it will cut the cone in a rectangular hyperbola.

2. Given a cone and a point inside it determine the conics which have the given point as focus.

[Draw an elevation  $a'v'b'$  on a plane parallel to the plane containing the axis of the cone and the given point, and let  $f'$  be the elevation of the given point. The vertical traces of the required planes of section must be tangents at  $f'$  to the circles touching  $a'v'$  and  $b'v'$ , and passing through  $f'$ . Two solutions are generally possible.]

3. Shew that all sections of a right cone, made by planes parallel to tangent planes of the cone, are parabolas, and that the foci lie on a cone having with the first a common vertex and axis.

[Shew that the foci of parallel sections lie on a straight line through the vertex.]

4. Find the least angle of a cone from which it is possible to cut an hyperbola, whose eccentricity shall be the ratio of two to one.

5. Cut from a right cylinder an ellipse whose eccentricity shall be the ratio of the side of a square to its diagonal.

[In the cylinder inscribe a sphere, centre  $C$ ; determine a point  $X$  in the horizontal plane through the centre such that  $\frac{r}{CX}$  = the above ratio, where  $r$  is the radius of the sphere. The required plane of section must be a tangent plane to the sphere through the point  $X$ .]

6. Shew how to cut from a given cone a hyperbola whose asymptotes shall contain the greatest possible angle.

[The plane of section must be parallel to the axis, pp. 246 and 241.]

7. Cut from a given cone the hyperbola of greatest eccentricity.

[The plane of section must be parallel to the axis, p. 248.]

8. Different elliptic sections of a right cone are taken having equal major axes; shew that the locus of the centres of the sections is a spheroid, oblate or prolate, according as the vertical angle of the cone is greater or less than  $90^\circ$ .

[Consider a series of sections perpendicular to a principal section of the cone. The centre is a fixed point on a line of constant length (the major axis), sliding between two fixed lines (the two generators of that section). It therefore traces out an ellipse which by revolution round the axis of the cone generates a spheroid.]

9. Different elliptic sections of a right cone are taken such that their minor axes are equal; shew that the locus of their centres is the surface formed by the revolution of an hyperbola about the axis of the cone.

[Consider a series of sections perpendicular to a principal section of the cone. Take any section parallel to the base and divide the diameter of that section, so that the product of the two parts  $= b^2$  where  $b$  is the semi-length of the constant minor axis; the corresponding elliptic section must pass through this point of division, and all these points lie on a hyperbola, the asymptotes of which are the generators of the principal section taken (Prop. 1, p. 160).]

10. Shew how to cut a right cone so that the section may be an ellipse whose axes are of given lengths.

[The centre of the section made by the plane perpendicular to any principal section must be the intersection of the ellipse and hyperbola in which such principal section cuts the surfaces referred to in examples 8 and 9.]

11. Shew how to cut from a right cone a section of given latus rectum.

[Any point  $F$  on a hyperbola described as in Ex. 9 may be taken as focus, and the plane of section must be a tangent plane at  $F$  to the sphere inscribed in the cone, and passing through  $F$ .]

## CHAPTER X.

### CYCLOIDAL CURVES.

WHEN one curve rolls without sliding upon another, any point invariably connected with the rolling curve describes another curve, called a *roulette*. The curve which rolls is called the *generating curve*, and the curve on which it rolls is called the *directing curve*, or the *base*.

Only a few of the simpler examples of roulettes are here given, the first being the most simple of all, viz. the cycloid.

DEF. The cycloid is the path described by a point on the circumference of a circle, rolling upon a fixed right line, in one plane passing through the line.

In the construction this plane coincides with the plane of the paper.

PROBLEM 138. *To describe a cycloid, the diameter of the circle being given* (Fig. 143).

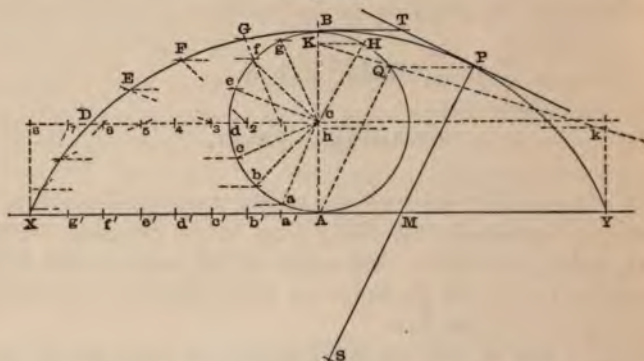
Let  $AB$  be the diameter of the given circle,  $C$  its centre, and suppose that the tracing point is the point  $B$ , and that at the moment  $A$  is the point of contact of the circle with the directing line. Draw the directing line  $XAY$  a tangent at  $A$  to the circle. The tracing point  $B$  will evidently reach the guiding line at points  $X$  and  $Y$  on opposite sides of  $A$  such that  $AX = AY =$  the semi-circumference  $AB$ , since each point of the semi-circumference comes down successively on a corresponding point of the line.

The following geometrical construction gives an exceedingly close approximation to the length of the circumference of a circle:—From  $C$ , the centre, draw a radius  $CH$  making an angle of  $30^\circ$  with the radius  $CB$ , and draw  $HK$  perpendicular to  $AB$  meeting it in  $K$ . At  $A$ , the extremity of the diameter through  $B$ , draw a



tangent to the circle and on it make  $AL = 3 \cdot AB$ .  $KL$  will be

Fig. 143.



very nearly the circumference of the circle and its semi-length may be taken for the length  $AX$  or  $AY$ .

[In the figure  $L$  does not fall within the limits of the paper, but if  $AK$  is bisected in  $h$  and  $hk$  on a parallel to the tangent at  $A$  be made  $= 3$  times the radius of the circle,  $Kk$  may be taken as the semi-circumference.]

Divide up  $AX$  into any number of equal parts (say 8) as at  $a', b', c', \dots$  and divide the semi-circumference  $AB$  into the same number as at  $a, b, c, \dots$ . Draw a line through  $C$  parallel to  $AX$ , which will evidently be the path of the centre of the circle, i. e. as the circle rolls along  $AX$  the centre will always be on this line; and draw  $a'1, b'2, c'3, \dots$  perpendicular to  $AX$ , the points 1, 2, 3, &c., being on the path of the centre. The point  $a$  will evidently come down to  $a'$ ,  $b$  to  $b'$ , and so on; and when  $a$  has come to  $a'$ , the centre of the circle will be at 1 and the tracing point will be on a line making an angle with  $a'1$  equal to the angle  $aCB$ , which is of course equal to  $ACg$ , since  $Aa = gB$ . Draw  $1G$  parallel to  $Cg$  and make  $1G = Cg$ , the radius of the rolling circle.  $G$  will be a point on the required curve.

Similarly, when  $b$  has rolled down to  $b'$ , the centre of the circle will be at 2 vertically above  $b'$ , the tracing point will be on a line making with  $b'2$  an angle  $=$  the angle  $bCB$ , i. e.  $=$  the angle  $ACf$ , or

it will be on a line  $2F'$  parallel to  $Cf$  and at a distance from 2 equal to the radius of the circle.

Similarly for the remaining points  $c', d', \&c.$

It will be noticed that the lengths  $1G, 2F, \&c.,$  may be determined without actual measurement by drawing through  $g, f, \&c.,$  parallels to  $AX$  meeting the corresponding lines through 1, 2,  $\&c.,$  in the points  $G, F, \&c.,$  the figures  $1CgG, 2CfF$  are parallelograms and therefore in each case  $1G = Cg, 2F = Cf,$  and so on.

The curve should be drawn free-hand through the series of points thus found, and the half loop corresponding to the circle rolling on  $AY$  may be found by the same construction or may be put in by symmetry. The line  $X8$  is a tangent to the curve at the point  $X$ .

The length  $AX$  may be determined arithmetically by multiplying the length of the radius  $AC$  by 3.14... and may then be laid down by scale: the diagonal scales usually supplied with cases of mathematical instruments can conveniently be used for the purpose. In many works on geometry the length  $AX$  is determined by dividing up the semi-circle into any number of equal parts (say  $n$ ) and laying off along  $AX$  the length of the chord of one of the parts repeated  $n$  times. This method is radically bad and should never be adopted: if the number of equal parts into which the semi-circle is divided is small it gives only a very rough approximation to the truth, while if the number is increased it is almost impossible to measure the length of the chord so accurately but that in repeating it  $n$  times an appreciable error will be introduced. A long length should in fact never be determined as the sum of a series of short ones.

*To draw the normal at any point of a cycloid.*

In all roulettes the normal at any point passes through the corresponding point of contact of the rolling and guiding curves. This point is called the *Instantaneous Centre*. The *direction of motion* of the tracing point will evidently at any moment be perpendicular to the line between it and the point about which the rolling curve is turning, i.e. the corresponding instantaneous



centre, and since the direction of motion at any point must coincide with the tangent at that point, the normal must pass through the instantaneous centre.

In the figure, when the tracing point is at  $E$  the centre is at 3 and  $c'$  is the instantaneous centre, so that  $Ec'$  is the normal at  $E$ ; this is evidently parallel to  $eA$ ,  $e$  being the point in which a parallel to  $AX$  through  $E$  meets the circle on  $AB$  as diameter, so that the normal at any point  $P$  may be thus constructed:—

Through  $P$  draw a parallel to the directing line  $AY$  meeting the circle on  $AB$  in the point  $Q$ . The normal at  $P$  will be parallel to  $AQ$ , and since the angle  $AQB$  is a right angle the tangent at  $P$  will be parallel to  $QB$ .

If the normal at  $P$  meet the directing line in  $M$  and  $PM$  be produced to  $S$  so that  $PS = 2PM$ ,  $S$  will be the centre of curvature at the point  $P$ . The evolute of the cycloid is two equal semi-cycloids, the vertices being at  $X$  and  $Y$  and the cusp on  $BA$  produced at a distance from  $A = AB$ .

Let the tangent at  $P$  meet the tangent at the vertex in  $T$ , then the length of the arc  $BP$  of the cycloid is double the intercept  $TP$  of the tangent, i.e. double the chord  $BQ$  of the circle. Hence the whole length of the cycloid is 4 times the diameter of the generating circle.

DEF. If, as in the cycloid, a circle rolls along a straight line, any point in the plane of the circle but not on its circumference traces out a curve called a *Trochoid*.

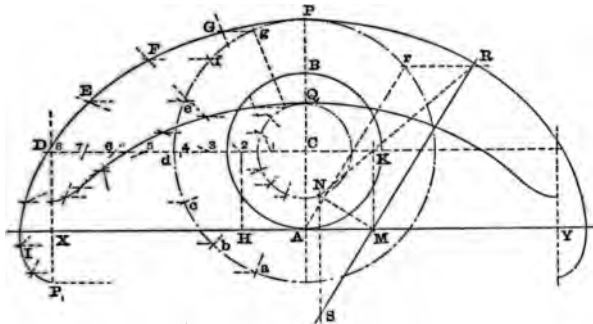
PROBLEM 139. *To describe a trochoid, the diameter of the circle and the distance of the tracing point from its centre being given (Fig. 144).*

Let  $AB$  be the diameter of the given circle,  $C$  its centre, and  $CP$  the distance of the tracing point from the centre.

Draw  $XAY$  a tangent to the circle, and as in the last problem determine the length  $AX$  or  $AY$  equal to the semi-circumference of the circle  $AB$ . Draw  $CS$ , the path of the centre, through  $C$  parallel to  $XAY$ , and through  $X$  draw  $X8$  perpendicular to  $XA$ .

Divide  $C8$  into any number of equal parts (8 in the fig.), and with centre  $C$  and radius  $CP$  draw a circle. The point  $P$  in which this

Fig. 144.



circle cuts  $AB$  produced will be the vertex of the required curve. Divide the semi-circumference of the circle into the same number of equal parts  $Pg$ ,  $gf$ , &c., as has been chosen for the division of the path of the centre.

Draw  $1G$  parallel to  $Cg$  and  $gG$  parallel to  $AX$ : their intersection  $G$  will be a point on the required curve. Similarly  $2F$  parallel to  $Cf$  and  $fF$  parallel to  $AX$  will intersect in a point on the curve, and so on in succession. When  $B$  has come down to  $X$  the tracing point will evidently be at  $P_1$  vertically below  $X$  on  $AX$  produced so that  $8P_1 = CP$ ; the tangent at  $P_1$  is parallel to  $AX$ .

The construction is obvious from that of the cycloid.

In the figure a second trochoid is drawn generated by a point  $Q$  inside the rolling circle, to which the foregoing description applies exactly by the substitution of  $Q$  for  $P$ .

*To draw the normal at any point of a trochoid.*

Consider for a moment the point  $F$ . When the tracing point is at  $F$  the centre of the rolling circle will be at 2 and the point of contact of the rolling circle and directing line will be  $H$  on  $AX$  vertically below 2; i.e.  $H$  will be the instantaneous centre, and therefore  $FH$  will be the normal at  $F$ , since the direction of motion of  $F$  must be perpendicular to  $FH$ . But  $FH$  is parallel to  $fA$ ,

since the triangles  $F2H, fCA$  are in all respects equal and are similarly situated, and therefore the normal at any point  $R$  may be thus constructed :—

Through  $R$  draw a parallel to the directing line meeting the circle described with  $C$  as centre and  $CP$  as radius in the point  $r$ , and the normal  $RM$  will be parallel to the line joining  $r$  to  $A$ , the lowest point of the rolling circle when its centre is  $C$ .

*To find the centre of curvature at any point  $R^*$ .*

Find  $K$ , the position of the centre of the rolling circle corresponding to  $R$ . ( $K$  will of course be vertically above  $M$ .) Join  $RK$  and draw  $MN$  perpendicular to  $RM$  meeting  $RK$  in  $N$ . Draw  $NS$  perpendicular to the guiding line meeting  $RM$  in  $S$ .  $S$  will be the required centre of curvature.

DEF. The Epicycloid is the path described by a fixed point on the circumference of a circle rolling on the convex side of a fixed circle, both circles lying in the same plane.

PROBLEM 140. *To describe an epicycloid, the radii of the rolling and directing circles being given (Fig. 145).*

Let  $O$  be the centre of the directing circle,  $OA$  its radius,  $AC$  the radius of the rolling circle,  $C$ , on  $OA$  produced, its centre, and let  $B$  be the other extremity of the diameter through  $A$ . Suppose  $B$  to be one position of the tracing point. As the one circle rolls round the other let the point  $B$  come down to  $X$  on the one side of  $A$  and to  $Y$  on the other,  $X$  and  $Y$  being on the directing circle. The arc  $AX$  will necessarily be equal to the arc  $AY$ , and equal to the semi-circumference of the rolling circle.

These points may be thus determined :—

Let the length of the semi-circumference  $AB$  be  $S$ , then

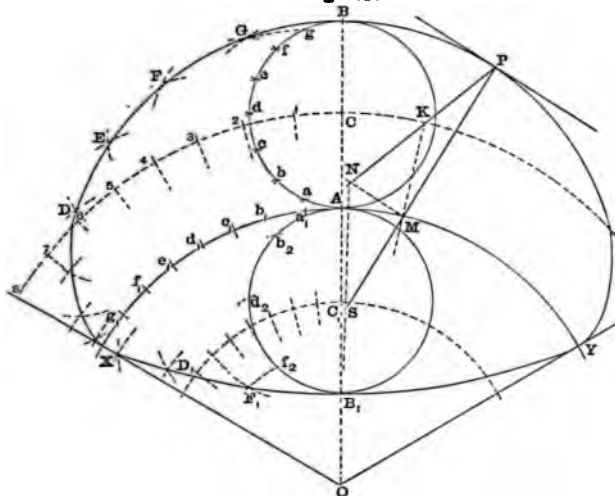
$$S = \pi \cdot AC,$$

$\pi$  being the circular measure of two right angles.

\* The construction for the centre of curvature of this and the following roulettes was given by M. Savary in his *Leçons des Machines à l'École Polytechnique*, and is quoted by Williamson, *Differential Calculus*, 3rd ed., p. 345, where its proof is given.

Let  $\theta$  be the circular measure of the angle subtended by the

Fig. 145.



arc  $AX$  (the length of which is  $S$ ), at the centre of the directing circle; then

$$S = \theta \cdot AO = \pi \cdot AC;$$

$$\therefore \theta : \pi :: AC : AO,$$

or if  $n$  is the number of degrees in the angle  $AOX$ ,

$$n : 180^\circ :: AC : AO,$$

$$\text{or } n = 180^\circ \frac{AC}{AO},$$

which determines  $n$ .

[In the figure  $AO = 3AC$  so that the angle  $AOX$  contains  $60^\circ$ .]

Draw the path of the centre of the rolling circle, i.e. an arc with centre  $O$ , and radius  $OC$ , and let  $OX$  produced meet it in 8. Divide up the arc  $C8$  into any convenient number of equal parts (8 in the fig.) and draw the radii  $O1, O2, \dots$  cutting the directing circle in  $a' b' \dots$ . Divide up the semi-circumference of the rolling circle into the same number of equal parts  $Aa, ab, \dots$ :



As the one circle rolls on the other, the point  $a$  will evidently come down to the point  $a'$ ,  $b$  to  $b'$  and so on: when  $a$  has come to  $a'$ , the centre of the rolling circle will be at the point 1, and the tracing point will evidently be on a line making with  $a'1$  an angle equal to the angle  $aCB$  which is equal to the angle  $ACg$ . Hence an arc described with centre 1, and radius  $CB$ , will intersect an arc described with centre  $O$ , and radius  $Og$ , in a point  $G$  of the required curve, for the triangles  $G1O$  and  $gCO$  are equal in all respects:—i.e.  $G$  is the position of the tracing point corresponding to  $a'$ , being the point of contact of the rolling and directing circles.

Similarly an arc described with centre 2, and radius  $CB$  will intersect an arc described with centre  $O$ , and radius  $Og$  in a point  $F$  of the required curve, and so on in succession for the points 3, 4, &c.

The arcs  $gG, fF$ , &c. will cut the corresponding arcs described with the successive centres 1, 2, &c. in two points, but it is evident by inspection which of the points must be taken, viz. that on the side of the corresponding radius  $O1, O2$ , &c. remote from  $OA$ .

The radius  $OX8$  is a tangent to the curve at the point  $X$ .

*To draw the normal at any point  $P$  of an epi-cycloid.* From  $P$  with the radius  $AC$  of the rolling circle describe an arc cutting the path of the centre in  $K$ . [It will do so in two points but the one lying within the angle  $POB$  must be taken.] This will be the position of the centre of the rolling circle when the tracing point is at  $P$ . Draw  $KO$  cutting the directing circle in  $M$ , the point of contact between the circles when the tracing point is at  $P$ : i.e.  $M$  is the *instantaneous centre* corresponding to  $P$ .

Therefore  $PM$  is the normal at  $P$ .

*To find the centre and radius of curvature at any point  $P$ .* From  $M$  the *instantaneous centre* draw  $MN$  perpendicular to  $PM$  meeting  $PK$ , the radius of the rolling circle when the tracing point is at  $P$ , in  $N$ . Then  $NO$  ( $O$  being the centre of the guiding

circle) will cut  $PM$  produced in  $S$  the required centre of curvature.

DEF. The Hypo-cycloid is the path described by a fixed point on the circumference of a circle rolling on the concave side of a fixed circle, both circles lying in the same plane.

PROBLEM 141. *To describe a hypo-cycloid the radii of the rolling and directing circles being given (Fig. 145).*

$OA$  is the radius of the directing circle, and  $O$  its centre,  $AC'$  is the radius of the rolling circle, and  $B'$  the tracing point when the centre is at  $C'$ . The construction is identical with that for the epi-cycloid. In the figure the radius  $AC'$  is equal to  $AC$  the radius of the epi-cycloid, and  $B'$  of course reaches the directing line at  $X$  and  $Y$ —the points  $F'$  and  $D'$  are the positions of the tracing point when the points  $b$ , and  $d$ , are the points of contact of the rolling and directing circles.

DEF. When, as in the epi-cycloid, a circle rolls on the convex side of another, any point in the plane of the rolling circle, but not on its circumference traces out a curve called an *Epi-trochoid*.

PROBLEM 142. *To describe an epi-trochoid, the rolling and guiding circles, and the position of the tracing point being given (Fig. 146).*

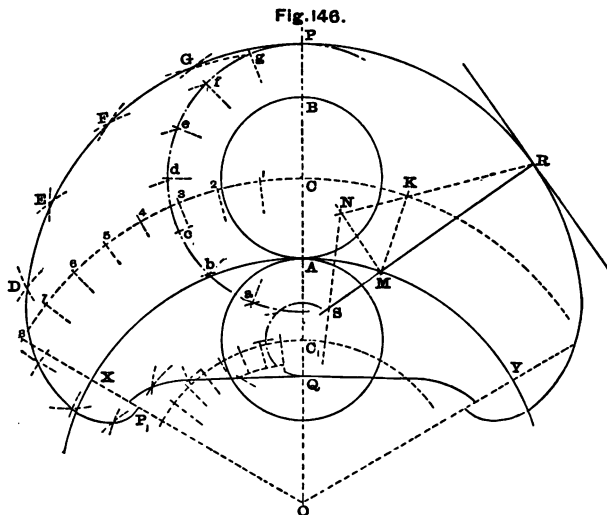
[In the figure the tracing point is assumed outside the rolling circle; it might be inside it.]

Let  $O$  be the centre of the directing circle,  $OA$  its radius,  $AC$  the radius of the rolling circle;  $C$ , on  $OA$  produced, its centre; let  $B$  be the other extremity of the diameter through  $A$ , and  $P$  on  $CB$  produced be one position of the tracing point. As in the epi-cycloid determine an arc  $AX$  or  $AY$  of the guiding circle equal in length to the semi-circumference of the rolling circle, so that  $B$  comes down to  $X$  and  $Y$  as the circle rolls round: i.e. construct angles  $AOX$  and  $AOY$  each containing  $n$  degrees where

$$n = 180^\circ \frac{AC}{AO}.$$

[In the figure  $AO = 3AC$  so that  $n = 60$ .]

Draw the path of the centre of the rolling circle, i.e. the circular arc with centre  $O$ , and radius  $OC$ , and produce the



radius  $OX$  to meet it in 8. Divide up  $C8$  into any convenient number of equal parts  $C1, 12, \&c.$ —(8 in the figure), and divide up the semi-circle drawn through  $P$ , with centre  $C$ , into the same number  $Pg, gf, \&c.$  With centre 1, and radius equal to  $CP$ , describe an arc, and with centre  $O$ , and radius  $Og$ , describe a second arc cutting it in  $G$ .  $G$  will be a point on the curve. Similarly with centre 2, and radius equal to  $CP$ , describe an arc, and with centre  $O$ , and radius  $Of$ , describe a second arc cutting it in  $F$ .  $F$  will be a point on the curve, and so on in succession for the points 3, 4, &c.

The arcs  $gG, fF, \&c.$  will cut the corresponding arcs described with the successive centres 1, 2, &c. in two points, but it is evident by inspection which of the points must be taken, viz. that on the side of the corresponding radius  $O1, O2, \&c.$  remote from  $OA$ .

The radius  $OX8$  is a normal to the curve at the point  $P_1$ .

*To draw the normal at any point  $R$  of an epi-trochoid.*

Find  $K$  the corresponding position of the centre of the rolling circle, i.e. with centre  $R$ , and radius equal to  $CP$ , describe an arc cutting the path of the centre in  $K$ . [It will do so in two points, but the one must be taken lying within the angle  $ROB$ .]

Draw  $KO$  cutting the directing circle in  $M$ .  $M$  will be the *instantaneous centre* corresponding to  $R$ . Therefore  $RM$  is the normal at  $R$ .

*To find the centre and radius of curvature at any point  $R$ .*

From  $M$  the *instantaneous centre* draw  $MN$  perpendicular to  $RM$  meeting  $RK$  ( $K$  being as above) in  $N$ . Then, if  $O$  is the centre of the directing circle,  $ON$  will cut the normal  $RM$  produced in  $S$ , the required centre of curvature.

**DEF.** The *Hypo-trochoid* is the curve traced out by any point in the plane, but not on the circumference of a circle, rolling on the concave side of a fixed circle, both circles lying in the same plane.

**PROBLEM 143.** *To describe a hypo-trochoid, the directing and rolling circles, and the position of the tracing point being given (Fig. 146).*

[In the figure the tracing point is inside the rolling circle, but by the above definition this is not a necessary condition.]

$OA$  is the radius of the directing circle, and  $O$  its centre,  $AC'$  is the radius of the rolling circle, and  $Q$  the tracing point when the centre is at  $C'$ . The construction is identical with that for the epi-trochoid.

*Companion to the cycloid.*

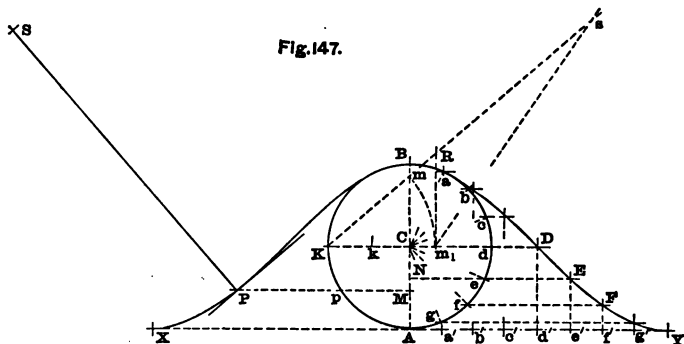
**DEF.** If a line  $NE$  (Fig. 147) be drawn perpendicular to a fixed diameter  $AB$  of a circle, meeting it in  $N$ , and the circle itself in  $e$ , and if  $NE$  be made equal to the arc  $Be$ , the locus of the point  $E$  is called the *Companion to the Cycloid*.

**PROBLEM 144.** *To describe the companion to the cycloid, the generating circle being given (Fig. 147).*

$C$  is the centre, and  $AB$  a diameter of the given circle. Through



*A* draw  $XAY$  a tangent to the given circle, and on it make  $AX = AY =$  the semi-circumference. (Prob. 138.) Divide  $AY$



into any convenient number of equal parts as at  $a', b', c' \dots$  and divide the semi-circumference  $AB$  into the same number of equal parts as at  $a, b, c \dots$

It will be observed that the lettering proceeds from  $A$  in the one case, and from  $B$  in the other.

Through  $a', b', c' \dots$  rule perpendiculars to  $AY$ , and through  $a, b, c \dots$  rule parallels to  $AY$ . The intersections of corresponding lines as  $D, E, F \dots$  are points on the required curve.

The construction is obvious.

*To draw the tangent at any point P.*

Draw  $PM$  parallel to  $AX$  meeting the circle in  $p$ , and the diameter  $AB$  in  $M$ . Make  $Cm$  on  $CB = Mp$ , and join  $m$  to  $K$  the extremity of the diameter perpendicular to  $AC$ . The tangent at  $P$  is parallel to  $mK$ . The curve has parallel tangents at points equi-distant from  $CK$ .

*To find the radius of curvature at any point P.*

It is easily proved analytically that  $\rho = \frac{mK^3}{a \cdot CM}$ , where  $\rho$  is the radius of curvature,  $m$  and  $M$  are points corresponding to  $P$  as above, and  $a$  is the radius of the generating circle.

Make  $Km_1$  on  $KC = Km$ , and draw  $m_1R$  perpendicular to  $KC$  meeting  $Km$  in  $R$ , also make  $Kk$  on  $KC = CM$ . Through  $m_1$  draw  $m_1s$  parallel to  $kR$  meeting  $KR$  in  $s$ , and  $Ks$  will be the length of the required radius of curvature. Make  $PS$  on the normal at  $P = Ks$ , and  $S$  will be the centre of curvature at  $P$ .

Evidently  $Ks : KR :: Km_1 : Kk$ ,

or 
$$Ks = \frac{KR \cdot mK}{CM},$$

but  $KR : Km :: Km_1 : CK$ , or  $KR = \frac{mK^2}{a}$  ;

$$\therefore Ks = \frac{\overline{mK}^3}{a \cdot CM} = \rho.$$

#### EXAMPLES ON CHAPTER X.

1. Shew that if the diameter of the rolling circle be half that of the directing circle, the hypo-cycloid becomes a straight line.

2. Shew that if the diameter of the rolling circle be half that of the directing circle any hypo-trochoid becomes an ellipse.

3. Shew that if  $AOB$  be a diameter of the guiding circle, and  $P$  any point on it, the hypo-cycloids described by the circles having  $AP$  and  $BP$  as diameters, and  $P$  as tracing point, are identical.

4.  $A$  is a fixed point on the circumference of a circle of radius  $R$ . The points  $L$  and  $M$  are taken on the same side of  $A$  such that arc  $AL = m \cdot$  arc  $AM$ , where  $m$  is a constant. Shew that  $LM$  will always touch the epi-cycloid described with a circle of radius  $r \left( = \frac{R}{m+1} \right)$  rolling on a circle of radius  $\rho = R - 2r$ , the point  $A$  being the centre of the loop, and the centre of the guiding circle coinciding with that of the given one.

[As a numerical example take  $R = 3\frac{3}{4}$ ,  $m = 4$ .]

5.  $A$  is a fixed point on the circumference of a circle of radius  $R$ . The points  $L$  and  $M$  are taken on opposite sides of  $A$ , such that  $\text{arc } AL = m \cdot \text{arc } AM$ , where  $m$  is a constant. Shew that  $LM$  will always touch the hypo-cycloid described with a circle of radius  $r = \frac{R}{m-1}$  rolling under a circle of radius  $\rho = R + 2r$ , the point  $A$  being the centre of the loop and the centre of the guiding circle coinciding with that of the given one.

6. Shew that the radius of curvature of an epi-cycloid at any point varies as the perpendicular on the tangent at the point, from the centre of the fixed circle.

7. Shew that the evolute of the epi-cycloid described with guiding circle of radius  $a$  and rolling circle of radius  $b$  is a similar figure, the radii of the fixed and generating circles being  $\frac{a^2}{a-2b}$  and  $\frac{ab}{a+2b}$  respectively.

8. Shew that the evolute of the hypo-cycloid is a similar figure, the radii of the fixed and generating circles being  $\frac{a^2}{a-2b}$  and  $\frac{ab}{a-2b}$  respectively.

[To make a practicable figure  $b$  must be much smaller than  $a$ .]

9. If a parabola rolls on another equal parabola shew that the locus of the focus of the rolling one is the directrix of the other.

## CHAPTER XI.

### SPIRALS.

WHEN a line rotates in a plane about a fixed point of its length, and a point travels continuously in the same direction along the line according to some fixed law, the path of the moving point is called a spiral. The fixed point is called the *pole*; a fixed line in the plane passing through the pole from which the position angle of the moving line may be measured is called the *initial line*, and the line drawn from the pole to any point of the curve is called the *radius vector* of that point.

After rotating through four right angles the revolving line comes back to the position it occupied at starting, but there is of course a different value for the length of the radius vector, and since the position angle may increase without limit, so too does the value of the radius vector. Spirals consequently extend to an infinite distance from the pole, and consist of a series of convolutions round it.

Cases of mathematical instruments usually contain a diagonal scale, the unit of which is half-an-inch, and on which lengths can be read to two places of decimals. In the numerical examples which follow, this scale is intended to be used.

DEF. In the *Spiral of Archimedes* the length of the radius vector is directly proportional to its position angle.

Let  $r$  be the length of the radius vector of any point,  $\theta$  the angle which it makes with the initial line; the above definition is expressed symbolically by the equation  $r = a\theta$ , where  $a$  is any numerical constant.



therefore

$$OQ - OP = a \times \text{circ. meas. of } QOP$$

or

$$a = \frac{OQ - OP}{\text{circ. meas. of } QOP}.$$

$OQ - OP$  can be measured by scale, the number of degrees in the angle  $QOP$  can be measured by a protractor and its circular measure can be obtained from a table of the circular measures of angles, and the numerical value of  $a$  thus calculated; then  $\theta = \frac{OP}{a}$ , the length  $OP$  being of course measured on the same scale as that used for determining  $OQ - OP$ , which gives the circular measure of the angle between  $OP$  and the initial line, and the corresponding number of degrees can be obtained from the table.

To take a numerical example:

Let the unit of length be  $\frac{1}{2}$  an inch. Suppose

$$OQ = 2,$$

$$OP = 1.5,$$

and the angle  $QOP = 60^\circ$ , the circular measure of which is

$$\frac{3.14159...}{3} = 1.0472...$$

$$a = \frac{2 - 1.5}{1.0472} = \frac{.5}{1.0472} = .477...$$

then

$$\theta = \frac{1.5}{.477...} = 3.14$$

the number of degrees corresponding to which may be taken  $180^\circ$ .

The initial line will therefore be the line  $OA$ . If the tracing point after one complete revolution of the generating line cuts  $OP$  again in  $P'$  we have

$$OP = a\theta$$

and

$$OP' = a(\theta + 2\pi),$$

therefore

$$OP' - OP = 2\pi a.$$

Successive points on the curve may at once be found thus:—  
Construct an angle  $QOR = \text{angle } QOP$ ; with centre  $O$  and radius  $OP$  describe an arc cutting  $OQ$  in  $p$ ; on  $OQ$  produced make

$Qr = Qp$  and with centre  $O$  and radius  $Or$  describe an arc cutting  $OR$  in  $R$  a point of the curve.

Similarly if  $R\hat{O}S = P\hat{O}Q$ , and  $Qs$  on  $OQ$  produced  $= 2Qp$ , an arc described with centre  $O$  and radius  $Os$  will cut  $OS$  in  $S$  a point of the curve. (In the figure  $S$  coincides with  $A$  on the initial line.)

In like manner points can be found nearer the pole than  $P$  by constructing angles on the side of  $OP$  remote from  $Q$  equal respectively to  $POQ$ ,  $2POQ$ ,  $3POQ$ , &c., and diminishing the radii vectors by the constant difference  $pQ$ .

*To draw the tangent at any point of the curve.*

A known expression for the angle which the tangent at any point makes with the radius vector is  $\phi = \tan^{-1} \frac{r}{a}$ , i.e. the tangent of the angle is the radius vector divided by the given constant of the curve.

Therefore to draw the normal at any point  $Q$ , on the radius  $OG$  at right angles to  $OQ$  measure a length  $OG = a$ , the constant of the curve, and  $QG$  will be the normal at  $Q$ , for evidently

$$\tan OGQ = \frac{OQ}{OG} = \frac{r}{a} = \tan \phi.$$

Hence if a circle be drawn with centre  $O$ , and radius  $= a$ , normals at all the points on the curve can at once be drawn by merely joining them to the corresponding points in which such circle cuts the perpendicular radii.

The initial line is a tangent at the pole.

If  $\rho$  is the radius of curvature at any point

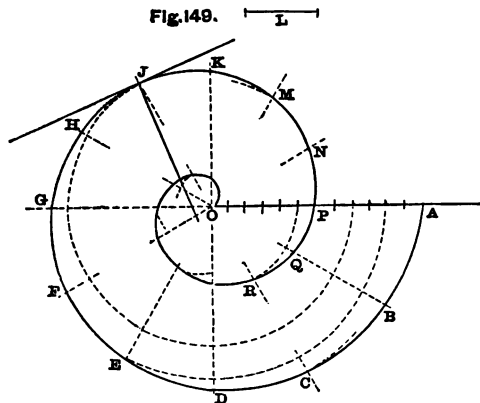
$$\rho : \sqrt{a^2 + r^2} :: a^2 + r^2 : 2a^2 + r^2,$$

so that  $\rho$  can be calculated without much difficulty.

PROBLEM 146. *To describe the spiral of Archimedes, the pole, the initial line and the constant of the curve being given (Fig. 149).*

Here  $a$  is given in the equation  $r = a\theta$ . Let  $O$  be the pole, and  $OA$  the initial line. In the figure, the unit being the length  $L$ ,  $a = .239$ .

Determine some convenient length of radius corresponding to a multiple ( $n$ ) of 4 right angles; say the greatest distance to which



it is proposed to draw the curve. In the figure e.g.  $A$  is taken at angular distances of 8 right angles from the initial line (i.e.  $n = 2$ ), so that

$$\begin{aligned} OA &= .239 \times 4\pi \\ &= .239 \times 4 \times 3.14159... \\ &= 3.60 \text{ units.} \end{aligned}$$

Draw  $OD$  at right angles to  $OA$  and divide up the quadrants formed at  $O$  into any number ( $m$ ) of equal parts (in the figure  $m = 3$ ) and draw the radii  $OB, OC$ , &c. through the points of division. Divide  $OA$  into  $4.m.n$  equal parts. In the figure therefore  $OA$  is divided into 24 equal parts. Then arcs drawn through the successive points on  $OA$  with centre  $O$  will intersect the corresponding radii in points on the curve. The point  $P$  in the figure of course bisects  $OA$ , and after one complete convolution has been found the curve can be completed by measuring from  $B, C$ , &c. on the successive radii a constant distance  $BQ, CR$ , &c. =  $AP$ .

#### THE RECIPROCAL OR HYPERBOLIC SPIRAL.

**DEF.** In this curve the length of the radius vector is inversely proportional to its position angle.



The equation to the curve may therefore be written  $\frac{1}{r} = a\theta$ , where  $r$  is the length of any radius vector,  $\theta$  the circular measure of the angle it makes with the initial line, and  $a$  a numerical constant.

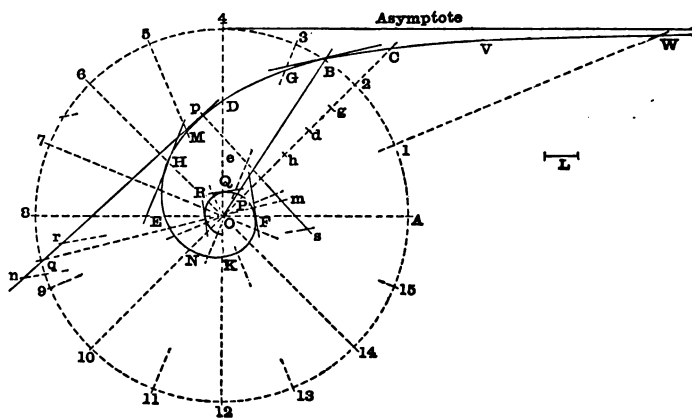
When  $\theta = 0$ ,  $r$  is therefore infinite, and  $r$  diminishes as  $\theta$  increases, but the curve does not reach the pole for any finite value of  $\theta$ . Corresponding to the value  $\theta = 1$ ,  $r = \frac{1}{a}$ ; i.e. the radius vector making  $57.2957\dots$  degrees with the initial line is  $\frac{1}{a}$  units long.

A line parallel to the initial line and  $\frac{1}{a}$  units distant from it, is an *asymptote* to the curve.

**PROBLEM 147.** *To draw the reciprocal spiral, the pole, the initial line and the unit and constant of the curve being given (Fig. 150).*

Let  $O$  be the pole and  $OA$  the initial line. In the figure  $a = \frac{1}{6}$  the unit being the length  $L$ .

Fig. 150.



Draw  $O4$  perpendicular to  $OA$  and with  $O$  as centre, and  $\frac{1}{a}$  as radius describe a circle 4, 8, 12,... and divide it up into any number of equal parts, as at 1, 2, 3...

Draw the line  $OB$  making  $57.2957...$  degrees with  $OA$  and cutting the circle in  $B$ ;  $B$  will be a point on the curve.

Determine the length of radius vector corresponding to any convenient division of the circle—say the radius making  $45^\circ$  with the initial line—i.e. determine

$$r = \frac{1}{a} \cdot \frac{4}{\pi} = \frac{24}{3.14159} = 7.63...$$

Draw the line  $O2$ , and produce it to  $C$  making  $OC = 7.63$  units.  $C$  will be a point on the curve. As the angle doubles the radius diminishes one half; so that if  $OC$  is bisected in  $d$ , the length  $Od$  will be the length of radius vector making a right angle with the initial line, i.e.  $D$  on the line  $O4$ ,  $OD$  being equal to  $Od$ , is another point on the curve.

Bisect  $OD$  in  $e$  and make  $OE$  on  $OB$ ;  $E$  will be a point on the curve.

$OE$  is also of course  $= \frac{1}{4}OC$ .

Similarly  $OF$  the radius corresponding to  $\theta = 2\pi$  is  $\frac{1}{2}OE$  or  $\frac{1}{8}OC$ .

$G$  the point on the curve corresponding to  $\theta = \frac{3}{2} \cdot \frac{\pi}{4}$  is at a distance  $\frac{2}{3}$  of  $OC$  from  $O$ .  $OH$  the radius corresponding to  $\theta = 3 \cdot \frac{\pi}{4}$  is of course  $\frac{1}{2}OG$  or  $\frac{1}{3}$  of  $OC$ .  $OK$  the radius corresponding to  $\theta = 6 \cdot \frac{\pi}{4}$  is  $\frac{1}{2}OH$ .

$OM$  the radius corresponding to  $\theta = \frac{5}{2} \cdot \frac{\pi}{4}$  is  $\frac{2}{5}OC$ , and  $ON$  the radius corresponding to  $\theta = 5 \cdot \frac{\pi}{4}$  is  $\frac{1}{2}OM$  or  $\frac{1}{5}OC$ .

In the second convolution

$OP$  on  $OC$ , i.e. corresponding to  $\theta = 9 \cdot \frac{\pi}{4}$  is  $\frac{1}{9}OC$ ,

$OQ$  on  $OD$  „ „ „ =  $10 \cdot \frac{\pi}{4}$  is  $\frac{1}{10}OC$ ,

$OR$  on  $OH$  „ „ „ =  $11 \cdot \frac{\pi}{4}$  is  $\frac{1}{11}OC$ ,

$OS$  on  $OE$  „ „ „ =  $12 \cdot \frac{\pi}{4}$  is  $\frac{1}{12}OC$ ,

and so on, and similarly any additional number of points can be obtained.

In the figure  $OV$  bisects the angle  $AOG$  and therefore

$$OV = 2 \cdot OG,$$

$OW$  bisects the angle  $AOC$  and  $OW = 2 \cdot OC$ .

*To draw the tangent at any point  $p$ .*

Draw the radius  $Oq$  of the circle described with centre  $O$  and radius  $\frac{1}{a}$  perpendicular to  $Op$ .  $pq$  will be the tangent at  $p$ .

*To determine the centre and radius of curvature at any point  $p$ .*

Draw the normal  $pm$  perpendicular to the tangent  $pq$  and meeting  $qO$  in  $m$ . On  $pq$  make  $pr = Oq = \frac{1}{a}$ , and  $pn = mq$ . Then  $ns$  drawn through  $n$  parallel to  $rm$ , meeting  $pm$  in  $s$ , determines  $s$  the required centre.

#### THE LITUUS.

In this curve the radius is inversely proportional to the square-root of the angle through which it has revolved. Its equation is therefore

$$\frac{1}{r} = a \sqrt{\theta},$$

or as it may also be written

$$\frac{1}{r^2} = a^2 \theta.$$

The radius therefore diminishes as the angle increases and is of infinite length when  $\theta = 0$ : it never vanishes however large  $\theta$  may be, so that the spiral never reaches the pole, but makes an infinite series of convolutions round it.

**PROBLEM 148.** *To draw the Lituus, the pole, the initial line and the unit and constant of the curve being given (Fig. 151).*

Let  $O$  be the pole, and  $OA$  the initial line. In the figure  $a = \frac{1}{3}$ , the unit being the length  $L$ . Draw  $OC$  perpendicular to  $OA$ , and determine the value of  $r$  corresponding to  $\theta = \frac{\pi}{2}$ , i.e. to  $\theta$  being the circular measure of a right angle.

In the figure

$$\frac{1}{OC^2} = \frac{1}{9} \cdot \frac{\pi}{2},$$

$$\text{or} \quad \overline{OC}^2 = 9 \times \frac{2}{3.14159 \dots} = 5.72.$$

Make  $Oc$  on  $OA$  equal to this length, and make  $OB$  on  $AO$  produced equal to unity on the scale adopted. A mean proportional between  $OB$  and  $Oc$  will evidently be the required length  $OC$ , i.e. a semi-circle on  $Bc$  will cut  $OC$  in  $C$ , a point on the curve.

Draw radii  $OG$ ,  $OH$  bisecting the quadrants  $COD$ ,  $DOE$ .

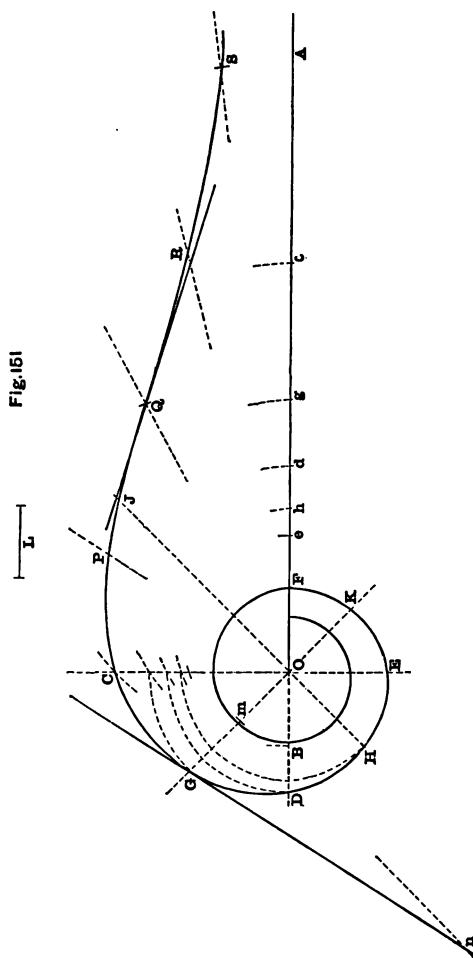
Trisect  $Oc$  in  $e$  and  $g$ , and take two parts measured from  $O$  as  $Og$ . A mean proportional between  $OB$  and  $Og$  will be equal to the length  $OG$  at which the curve cuts the bisector  $OG$  of the right angle  $COB$ .

Bisect  $Oc$  in  $d$ . A mean proportional between  $OB$  and  $Od$  will give the length of the radius vector  $OD$  corresponding to  $\theta = \pi$ .

Divide  $Oc$  into five equal parts, and take two of them from  $O$  as  $Oh$ . A mean proportional between  $OB$  and  $Oh$  will give the length  $OH$  of the radius vector corresponding to

$$\theta = \frac{5\pi}{4}.$$

A mean proportional between  $OB$  and  $Oe$  ( $\frac{1}{3}$ rd of  $Oc$ ) gives  $OE$  the length corresponding to  $\theta = \frac{3\pi}{2}$ .



Similarly a mean proportional between  $OB$  and  $\frac{2}{7}Oc$  would give  $OK$  the radius corresponding to  $\theta = \frac{7}{4}\pi$ , and a mean pro-

portional between  $OB$  and  $\frac{1}{4}Oc$  would give  $OF$  corresponding to  $\theta = 2\pi$ , but this is more easily determined by making it equal  $\frac{1}{2}OC$ , for since the square of the radius is inversely proportional to the angle, the radius diminishes  $\frac{1}{2}$  as the angle increases four times.

For the same reason the length  $OJ$  on  $HO$  produced will be  $2OD$  since the angle  $AOJ = \frac{1}{4}$  of two right angles.

Draw the angle  $AOP$  to contain  $57.29\dots$  degrees; the arc subtending this angle is equal to the radius, i.e. corresponding to it,  $\theta = 1$ , and therefore  $OP$  the corresponding radius must contain  $\frac{1}{a}$  units (in the figure  $OP = 3$ ).

Bisect the angle  $AOP$  by  $OQ$ , and make  $OQ^2 = \frac{2}{a^2}$  (in the figure  $OQ = \sqrt{18} = 4.24\dots$ ).

$Q$  is a point of contrary flexure in the curve, i.e. at that point it becomes convex towards the initial line, the radius of curvature being infinite.

Bisect the angle  $AOQ$  by  $OR$ , and make  $OR = \text{twice } OP$ ;  $R$  will be a point on the curve.

Bisect  $AOR$  by  $OS$ , and make  $OS = \text{twice } OQ$ ;  $S$  will be a point on the curve.

In the second convolution the following table gives the values of  $r$  corresponding to successive values of  $\theta$  differing by  $45^\circ$ , and similarly for the third convolution.

If 2 be taken as the numerator of all the fractions the successive denominators evidently differ by unity.

The values of  $r$  may of course all be calculated arithmetically instead of being obtained geometrically from the calculated value of one of them.

| $\theta$                | $r^2$             |                                                                        |
|-------------------------|-------------------|------------------------------------------------------------------------|
| $2\pi + \frac{\pi}{4}$  | $\frac{2}{9} Oc$  | i.e. $r$ must be a mean proportional between $OB$ and $\frac{2}{9} Oc$ |
| $2\pi + \frac{\pi}{2}$  | $\frac{1}{5} Oc$  | " " "                                                                  |
| $2\pi + \frac{3\pi}{4}$ | $\frac{2}{11} Oc$ | " " "                                                                  |
| $3\pi$                  | $\frac{1}{6} Oc$  | " " "                                                                  |
| $2\pi + \frac{5\pi}{4}$ | $\frac{2}{13} Oc$ | " " "                                                                  |
| $2\pi + \frac{3\pi}{2}$ | $\frac{1}{7} Oc$  | " " "                                                                  |
| $2\pi + \frac{7\pi}{4}$ | $\frac{2}{15} Oc$ | " " "                                                                  |
| $4\pi$                  | $\frac{1}{8} Oc$  | " " "                                                                  |

To draw the tangent at any point.

A known expression for the angle which the tangent at any point makes with the radius vector is

$$\phi = \tan^{-1} \left( -\frac{2}{r^2 a^2} \right);$$

$$\therefore \tan \phi = -\frac{2}{r^2 a^2} = -2\theta.$$

The value of  $\tan \phi$  for any point can therefore easily be calculated numerically, and the corresponding number of degrees obtained from a set of tables; the angle then being plotted by means of a protractor. The minus sign in the above expressions denotes that  $\phi$  is always greater than a right angle when measured on the  $\theta$  side of the radius. It becomes more and more nearly a right angle as the angle increases. At the point  $Q$  corresponding to

$$\theta = \frac{1}{2}, \phi = 135^\circ.$$

The tangent may be constructed geometrically, though not very conveniently, thus:—

$$\text{we have} \quad \tan \phi = -\frac{2}{r^2 a^2} = -\frac{2OP^2}{r^2},$$

where  $OP$  is the radius corresponding to unit angle. Determine a length  $l$  such that

$$OB : OP :: 2OP : l,$$

so that  $l = 2OP^2$ , since  $OB$  is unity on the scale adopted ;

$$\therefore \tan \phi = -\frac{l}{r^2}.$$

The value of  $r^2$  is known, because it is some definite fraction of  $Ac$ . At  $G$  on the curve for example it is the length  $Og$ . From the point at which the tangent is required measure any convenient fraction of the length  $r^2$  along the radius vector, from the extremity draw a line perpendicular to the radius, and measure on it the same fraction of the length  $l$ , and the required tangent will pass through the point thus obtained.

In the figure  $Gm$  is  $\frac{1}{4}Og$ , and  $mn$  is  $\frac{1}{4}l$ , then  $Gn$  is the tangent.

Owing to the rapid diminution of  $r^2$  as the angle increases the method very soon becomes impracticable.

If  $\rho$  is the radius of curvature at any point, and  $r$  the corresponding radius vector,

$$\rho : \sqrt{4 + a^4 r^4} :: r(4 + a^4 r^4) : 2(4 - a^4 r^4).$$

#### *The Logarithmic or Equiangular Spiral.*

In this spiral the radius increases in a geometric while the angle increases in an arithmetic ratio. The angle of revolution is therefore proportional to the *logarithm* of the length of the radius vector, whence it derives its first name; it is called *equiangular* because in it the tangent at any point makes a constant angle with the radius vector.

This constant angle is called the angle of the spiral.

The equation to the curve is generally expressed in the form

$$r = a^\theta,$$

where  $a$  is some constant on which the form of the curve depends. From it evidently

$$\log r = \theta \log a ;$$



and since the logarithm of 1 is 0,  $r$  must evidently be of unit length when  $\theta = 0$ , i.e. the curve must cut the initial line at unit distance from the origin.

If this condition is not fulfilled the equation to the curve is of the form  $r = ba^\theta$  where  $b$  is another constant, and in this form the initial line must be taken so that it cuts the curve at a distance  $b$  from the origin.

The known constant value  $\phi$  of the angle which the tangent at any point makes with the radius vector is

$$\phi = \tan^{-1} \frac{1}{\log_e a},$$

where  $e$  is the base of Napierian logarithms, i.e.  $\phi$  is an angle such that

$$\begin{aligned} \tan \phi &= \frac{1}{\log_e a} \\ &= \frac{\log_{10} e}{\log_{10} a}. \end{aligned}$$

The value of  $\log_{10} e$  is 0.43429448.

From the definition of the curve it follows that any radius vector is a mean proportional between the two at equal angular distances from it on opposite sides. This property gives the best method of constructing the curve geometrically when the pole and two points are given or determined.

PROBLEM 149. *To draw a logarithmic spiral, the value of the constant in the equation, and the unit of the curve being given (Fig. 152).*

Let the equation be  $r = 1.15|^\theta$ , the unit being the length  $L$ .

Take  $O$  the pole, and  $OA$  the initial line—the curve will cut this line in the point  $M$  at unit distance from  $O$ .

Suppose the revolving line to have made one complete revolution, so that it again coincides with  $OA$ ; the corresponding value of  $\theta$  will be the circular measure of four right angles;—

i.e.  $2\pi$  or  $2(3.14159\dots) = 6.28318$ .

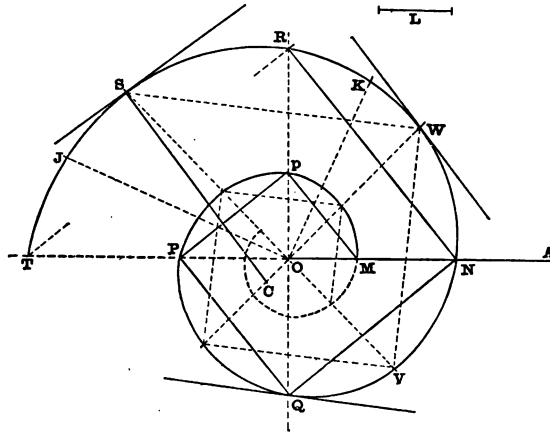
The corresponding value of  $r$  is given by

$$\begin{aligned}\log r &= 6.28318 \log (1.15) \\ &= 6.28318 \times .0606978 \\ &= .381376 ;\end{aligned}$$

$$\therefore r = 2.41 \text{ very nearly} = ON,$$

and  $N$  is a second point on the curve.

Fig. 152.



Make  $OP$  on  $MO$  produced a mean proportional between  $OM$  and  $ON$ , and  $P$  will be a third point on the curve. Through  $O$  draw  $QOp$  at right angles to  $OM$ , make  $OQ$  a mean proportional between  $OP$  and  $ON$ , and  $Q$  will be a point on the curve.

Similarly if the curve cuts  $QO$  again in  $R$ ,  $OR : ON :: ON : OQ$  which determines  $R$ . To do so evidently all that is necessary is to draw  $NR$  parallel to  $PQ$  or perpendicular to  $QN$ , and thus a series of points lying on two lines perpendicular to each other, and passing through the pole can be determined.

It is of course easy to interpolate points between those of the original series; for bisect the angle  $TOR$  by the line  $OS$ , and make  $OS$  a mean proportional between  $OT$  and  $OR$ , and on  $SO$  produced make  $OV$  a mean proportional between  $ON$  and  $OQ$ .

$S$  and  $V$  will be points on the curve.

Draw  $OW$  at right angles to  $OS$ , and make  $OW$  a mean proportional between  $OS$  and  $OV$  (i.e. on  $SV$  describe a semi-circle cutting  $OW$  in  $W$ ), and  $W$  will be a point on the curve. Then a series of points on the lines  $OS$  and  $OW$  can be obtained by drawing, as shewn by the dotted lines, parallels to  $SW$  and  $WV$  alternately.

The angle between any tangent and its radius vector is given by the equation

$$\phi = \tan^{-1} \frac{.43429448}{\log(1.15)},$$

$$\text{i.e. } \tan \phi = \frac{.43429448}{.0606978}$$

$$= 7.155,$$

whence  $\phi = 82^\circ$  nearly (more exactly  $81^\circ.58'$ ).

The tangents can therefore be drawn at all the points found by drawing lines through them making this angle with the radii.

The dotted part of the curve arises from negative values of the angle of rotation; it never reaches the pole.

#### *Centre of Curvature.*

The centre of curvature at any point  $S$  can easily be determined when the angle between the radius and tangent is known. Draw the normal  $SC$ , and from  $O$  the pole draw  $OC$  perpendicular to  $OS$  the radius vector;  $C$  will be the required centre.

**PROBLEM 150.** *To describe an equiangular spiral, the pole  $O$  and two points  $S$  and  $K$  on the curve being given (Fig. 152).*

Let  $OS = r_1$ ,  $OK = r_2$ , and the angle  $KOS = \alpha$ . (In the figure  $OS = 3.3$ ,  $OK = 2.78$ , and  $KOS = 1.22173\dots$  = the c.m. of  $70^\circ$ .)

The angle of the spiral may be determined from the following equation—

$$\tan \phi = \frac{\alpha \log_{10} e}{\log r_1 - \log r_2} = \frac{1.22173 \times .43429}{.5185139 - .4440448} = 7.124,$$

whence  $\phi = 82^\circ$  nearly.

The constant  $a$  of the curve is then given by

$$\log_{10} a = \frac{\log_{10} \epsilon}{\tan \phi} = \frac{.43429}{7.124} = .0609,$$

$\therefore a = 1.15$  very approximately.

Taking  $OK$  as the initial line the equation to the curve may be written

$$r = r_s a^{\theta}.$$

Draw  $OJ$  at right angles to  $OK$  and on it take a length  $OJ$  equal to  $r_s a^{\frac{\pi}{2}}$ , i.e. determined from the equation

$$\begin{aligned} \log OJ &= \log r_s + \frac{\pi}{2} \log a \\ &= .4440448 + 1.5707 \times .0609 = 3.47. \end{aligned}$$

We have now two points on radii at right angles to each other, and other points can at once be found by the preceding problem.

Any number of *points* on the curve can be found without determining either  $a$  or  $\phi$  by making each radius a mean proportional between the two at equal angular distances from it. Thus the radius bisecting the angle  $KOS$  must be a mean proportional between  $OK$  and  $OS$ , and the radius making an angle  $2a$  with  $OK$  must be a third proportional to  $OK$  and  $OS$ .

Points at equal angular distances can easily be found by Problem 8, when the lengths of two radii separated by that angular distance are known.

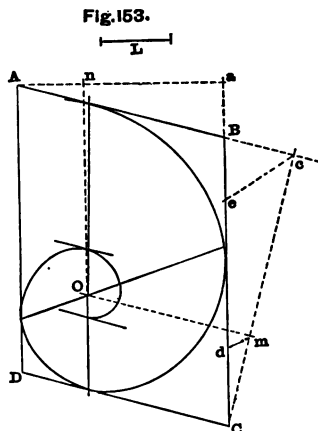
In practice  $\phi$  should always be determined, and tangents drawn at all the points found, because these tangents are of great assistance in tracing the curve through the points.

**PROBLEM 151.** *To inscribe a Logarithmic Spiral in a given parallelogram (Fig. 153).*

Let  $ABCD$  be the given parallelogram,  $a$  the circular measure of its acute angle. [In the figure  $AB = 3$ ,  $AD = 4$ , the unit being the length  $L$ , and the angle  $BAD$  contains  $75^\circ$ , so that its circular measure is  $1.309\dots$ ]

Let  $p$  and  $q$  be the perpendicular distances between the opposite pairs of sides,  $p$  being greater than  $q$ .

In the fig.  $p = 3.86$ , and  $q = 2.89$ .



If  $\phi$  be the angle of the spiral, it can be determined from the equation

$$\tan \phi = \frac{a \log_{10} e}{\log p - \log q},$$

or with the above dimensions

$$\begin{aligned} \tan \phi &= \frac{1.309 \times .43429}{.5865873 - .4608978} \\ &= \frac{.5385}{.1257} = 4.284, \end{aligned}$$

$\therefore \phi$  contains  $77^\circ$  very nearly.

Next determine the number ( $N$  suppose), the log. of which

$$= \frac{\pi \log e}{\tan \phi},$$

$$\begin{aligned} \text{i.e. in the present case } \log N &= \frac{3.1416 \times .43429}{4.284} \\ &= .3185, \end{aligned}$$

$\therefore$  from a table of logs  $N = 2.08$ .

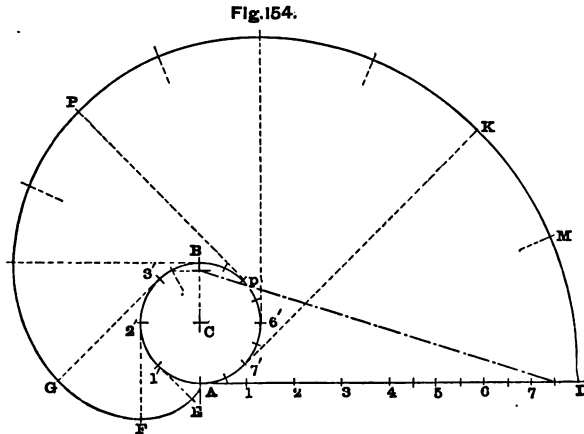
Divide the perpendiculars  $p$  and  $q$  so that one portion shall be to the other  $:: 1 : N$ , and lines drawn through these points of

division parallel to the sides of the given parallelogram will intersect in  $O$  the pole of the required spiral. In the figure the perpendicular  $Cc$  is divided by making  $Cd$  on  $CB =$  unity on any convenient scale, and  $de = 2.08$  on the same scale, then  $dm$  parallel to  $ec$  divides  $Cc$  at the point  $m$  in the required ratio. Similarly  $Aa$  is divided in  $n$ , and  $nO$  and  $mO$  perpendicular to  $Aa$  and  $Cc$  respectively intersect in  $O$  the required pole.

*The Involute of the Circle.* The Evolute of a Curve has already (p. 91) been defined as the locus of the centres of curvature, and considered with respect to its Evolute the curve is called the Involute of its Evolute. If an inextensible string be imagined to lie in contact with the evolute and to be kept stretched while gradually unwound from it, a certain fixed point on the string will describe the corresponding involute. The free portion of the string will be a tangent to the evolute at the point it quits it, and a normal to the involute at the point reached at the moment by the tracing point.

**PROBLEM 152.** *To draw the Involute of a given circle to pass through a given point (Fig. 154).*

1st. Let the given point be on the circle. Let  $C$  be the centre and  $AB$  a diameter of the given circle, and let  $A$  be a point



on the involute. Draw the tangent at  $A$ , and on it determine a length  $AD$  equal to the circumference (see p. 267). Divide  $AD$  into any convenient number of equal parts  $A1, 1.2, \dots$  &c., and the circumference into the same number  $A1', 1'2' \dots$ . Draw tangents to the circle at  $1', 2' \dots$

If we imagine a string unwound from the circle starting from  $A$ ,—when its point of contact is  $1'$ , i.e. when the free portion of the string is a tangent to the circle at  $1'$ , the length of the free portion will of course be equal to the arc  $A1'$ , or to the length  $A1$  of the straight line  $AD$ . Make  $1'E$  on the tangent at  $1'$  equal to  $A1$ , and  $E$  will therefore be a point on the curve. Similarly make  $2'F$  on the tangent at  $2'$  equal to  $A2$ , and  $F$  will be a point on the curve, and so on in succession.

2nd. Let the given point be  $P$ . Through  $P$  draw a tangent to the given circle meeting it in  $p$ . If  $A$  is the point where the required involute through  $P$  would meet the circle and  $\theta$  be the circular measure of the angle subtended at the centre by the arc  $Ap$  we have  $\theta = \frac{\text{arc } Ap}{AC}$ ; but the length of the arc  $Ap$  must be the line  $Pp$  so that if the lengths  $Pp$  and  $AC$  be measured on any scale the numerical value of  $\theta$  can be calculated, and the corresponding number of degrees obtained from a table. This of course determines  $A$  and the construction reduces to the first case.

As the distance from the pole increases and the points found on the curve get further and further apart, others can be determined between those of the original series by bisecting the corresponding arcs of the circle and divisions of the straight line  $AD$ , as shewn at  $M$ .

Tangents to the circle are of course normals to the involute, and the centre of curvature at any point is the point of contact of the tangent drawn from that point to the circle.

The involute of the circle is the locus of the intersection of tangents drawn at the points where any ordinate meets a circle and the corresponding cycloid.



EXAMPLES ON CHAPTER XI.

1. Draw a spiral of Archimedes to touch a given line, the pole  $O$  and the constant ( $a$ ), and unit of the curve being given.

[If  $r$  is the length of rad. vector to the point of contact of the given tangent, and  $p$  the length of the perpendicular on it from the pole

$$r^2 - \frac{p^2}{2} = \pm p \sqrt{a^2 + \frac{p^2}{2}}.$$

Construct therefore a rectangle equal to the sum of the two rectangles

$$p \times \frac{p}{2}, \text{ and } p \times \sqrt{a^2 + \frac{p^2}{2}}. \quad (\text{Prob. 18.})$$

The last expression is of course the length of the hypotenuse of a right-angled triangle, the sides of which are  $a$  and  $\frac{p}{2}$ , and is consequently always greater than  $\frac{p}{2}$ . The negative sign in the above equation therefore gives an imaginary result. A mean proportional between the sides of the rectangle constructed as above is the required length  $r$ .]

2. Draw a spiral of Archimedes to touch a given line  $PT$  at a given point  $P$ , and to have a given pole  $O$ .

[Through  $P$  draw  $Pa$  perpendicular to  $PT$ , and through  $O$  draw  $Oa$  perpendicular to  $OP$  meeting  $Pa$  in  $a$ . The length  $Oa$  is the unit of, and is proportional to, the constant of the curve, and the initial line is at an angle  $POA$  from  $OP$  given by circular measure of  $POA = \frac{OP}{Oa}$ .]

3. Draw a reciprocal spiral, the pole  $O$ , and two points  $P, Q$  on the curve being given.

[Compare problem 145. Let  $OP = r$ ,  $OQ = r_1$  of which let  $r$  be the greater; the angle  $POQ = a$ , and the angle between the initial



line and  $OP = \theta$ , then

$$\frac{1}{r} = a\theta,$$

$$\frac{1}{r_1} = a(\theta + a),$$

$$\therefore \frac{1}{r_1} - \frac{1}{r} = aa, \text{ or } a = \frac{1}{a} \cdot \frac{r - r_1}{rr_1}.$$

The value of  $a$  can be obtained from a table of the circular measures of angles, and if a fourth proportional  $l$  be determined to  $r - r_1$ ,  $r$  and  $r_1$

$$\frac{1}{a} = al,$$

which determines  $a$ . Any convenient scale can be used for measuring  $l$  and the unit of that scale will then be the unit of the curve; then  $\theta = \frac{1}{a} \times \frac{1}{r}$ , the length of  $r$  being measured on the same scale.

The initial line can then be drawn.]

4. Draw a reciprocal spiral, the pole  $O$ , a point  $P$  on the curve and the tangent at that point being given.

[Draw  $OT$  perpendicular to  $OP$  meeting the tangent at  $P$  in  $T$ .  $OT = \frac{1}{a}$ , so that the constant of the curve is known. If the circular measure of the angle between the initial line and  $OP$  is  $\theta$

$$\frac{1}{OP} = \frac{1}{OT} \cdot \theta, \text{ or } \theta = \frac{OT}{OP},$$

and the initial line can be drawn.]

5. Draw the Lituus, the pole  $O$  and two points  $P$  and  $Q$  on the curve being given.

[Let  $OP = r$ ,  $OQ = r_1$ ,  $r$  being greater than  $r_1$ ; the angle  $POQ = \alpha$ , and the angle between the initial line and  $OP = \theta$ .

$$\begin{aligned} \text{Then} \quad \frac{1}{r} &= a\sqrt{\theta}, \\ \frac{1}{r} &= a\sqrt{\theta + \alpha}, \end{aligned}$$

$$\therefore r^2 \theta = r_1^2 (\theta + \alpha),$$

$$\text{or} \quad \theta = \alpha \frac{r_1^2}{r^2 - r_1^2}.$$

Take a fourth proportional  $l$  to

$$r_1, r + r_1 \text{ and } r - r_1,$$

$$\text{then} \quad \theta = \alpha \frac{r_1}{l},$$

and can be calculated, the lengths  $r_1$  and  $l$  being measured on any convenient scale.]

6. Draw an equiangular spiral to touch three given lines  $AB$ ,  $BC$ ,  $CA$  in three given points  $P$ ,  $Q$ ,  $R$  respectively.

[On  $PR$  as chord describe a segment of a circle containing an angle equal to the external angle between the tangents  $AB$  and  $CA$ . This is a locus of the pole. Similarly on  $PQ$  as chord describe a segment of a circle containing an angle equal to the external angle between the tangents  $AB$  and  $BC$  which will be a second locus. The pole is thus determined.]

7. Draw an equiangular spiral of given angle  $(\phi)$  to touch three given lines  $AB$ ,  $BC$ ,  $CA$ .

[Suppose the spiral is to touch  $BA$  and  $BC$  produced. Through  $B$  draw a line dividing the angle  $ABC$  so that the perpendicular ( $p_1$ ) dropped from any point on it on  $AB$  is to the perpendicular ( $p_2$ ) dropped from the same point on  $BC$  as  $1 : \alpha^a$ , where  $\alpha$  is the constant of the required curve, and  $a$  is the circular measure of the supplement of the angle  $ABC$ , i.e.  $\frac{p_2}{p_1} = \alpha^a$ .

$\alpha$  is of course the number whose logarithm is

$$0.43429448 \times \cot \phi,$$

and can therefore be obtained from a table of logarithms. The line so drawn is a locus of the pole. Similarly draw a line through  $A$  dividing the angle between  $BA$  produced and  $AC$ , so that

$\frac{q_2}{q_1} = a^\beta$ , where  $q_1$  and  $q_2$  are perpendiculars on  $AB$ ,  $AC$  respectively, and  $B$  is the circular measure of the angle  $BAC$ . This line will be a second locus of the pole which is therefore known.]

8. Draw an equiangular spiral, the pole  $O$ , and two tangents  $TP$ ,  $TQ$  being given.

[Draw perpendiculars  $p_1$ ,  $p_2$  on  $TP$ ,  $TQ$  from  $O$  of which let  $p_2$  be the greater; then

$$\log a = \frac{\log p_2 - \log p_1}{\alpha} = \log_{10} e \cdot \cot \phi,$$

where  $a$  is the constant of the curve,  $\alpha$  the circular measure of the angle between the tangents alternate with that in which  $O$  lies, and  $\phi$  the constant angle between the tangent and radius vector.  $\phi$  can therefore be determined from a table of logarithms.]

## CHAPTER XII.

### MISCELLANEOUS CURVES.

#### *The Harmonic Curve or Curve of Sines.*

In this curve the ordinates are proportional to the sines of angles which are the same fractions of four right angles as the corresponding abscissæ are of some given length. It is the curve in which a musical string vibrates when sounded.

**PROBLEM 153.** *To draw the Harmonic Curve, the length and amplitude of a vibration being given (Fig. 155).*

Let  $AB$  be the given length,  $AO$  the given amplitude. With centre  $O$  on  $BA$  produced describe a semi-circle  $AA'$ , and divide it up into any convenient number of equal parts. Bisect  $AB$  in  $C$ , and divide up  $AC$  and  $CB$  into the same number of equal parts chosen for the semi-circle. Draw the successive ordinates  $1a$ ,  $1b$ , &c., and from the corresponding points on the semi-circle draw parallels to  $AB$  meeting the ordinates in  $a$ ,  $b$ ,... &c., which will be points on the curve. The length from  $A$  to  $C$  is half a wave length which will be repeated from  $C$  to  $B$  on the other side of  $AB$ .  $C$  is a point of inflection on the curve, the radius of curvature there becoming infinite.

*To draw the tangent at any point  $P$ .*

Through  $P$  draw  $pPM$  parallel to  $AB$ , cutting the semi-circle in  $p$ ; and make  $PM = AC$ . Draw  $pm$  perpendicular to  $OA$  cutting it in  $m$ , and make  $Mm'$  on  $MP = Om$ . Through  $M$  draw  $MN$  perpendicular to  $PM$  or  $AB$ , and on it make  $Mg = 3.14...$  on any convenient scale. On  $MP$  make  $Mk = \text{unity}$  on the same



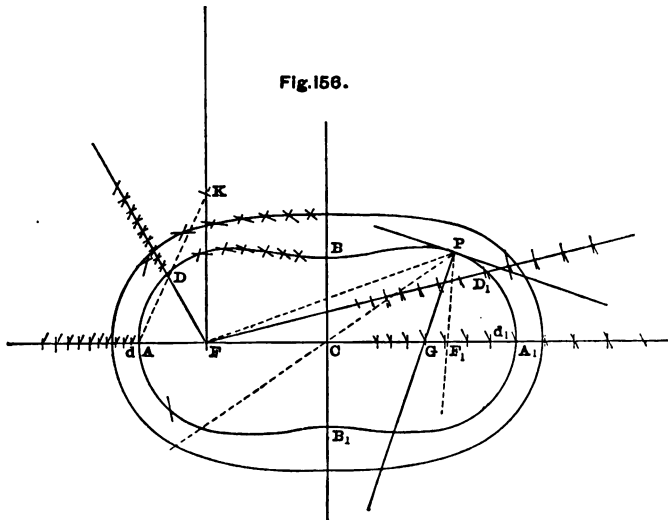
*Ovals of Cassini.*

When a point moves in a plane so that the product of its distances from two fixed points in the plane is constant, it traces out one of Cassini's ovals. The fixed points are called the foci. The equation of the curve is therefore  $rr_1 = k^2$ , where  $r$  and  $r_1$  are the distances of any point on the curve from the foci and  $k$  is a constant.

Corresponding to any given foci an infinite number of ovals may of course be drawn by varying  $k$ .

**PROBLEM 154.** *To describe an oval of Cassini, the foci  $F$  and  $F_1$  and the constant  $k$  of the curve being given (Fig. 156).*

Draw a line through  $F$  and  $F_1$  and bisect  $FF_1$  in  $C$ : through  $C$  draw  $BCB_1$  perpendicular to  $FF_1$ , and with  $F$  as centre and



radius  $= k$  describe an arc cutting  $BCB_1$  in  $B$  and  $B_1$ .  $B$  and  $B_1$  will evidently be points on the curve.

Draw  $FK$  perpendicular to  $FF_1$  and make  $FK = k$ , and on  $CF$  make  $CA$  and  $CA_1$  each  $= CK$ .  $A$  and  $A_1$  will be points on the

curve, for

$$CA^2 = CK^2 = CF^2 + FK^2,$$

$$\therefore CA^2 - CF^2 = k^2 = (CA + CF)(CA - CF);$$

but

$$CA + CF = F_1A \text{ and } CA - CF = FA,$$

$$\therefore FA \cdot F_1A = k^2.$$

With centre  $F$  and any radius greater than  $FA$  and less than  $FA_1$ , describe an arc  $dD$  cutting  $FA$  in  $d$ . Through  $K$  draw  $Kd_1$  perpendicular to  $dK$  and cutting  $FF_1$  in  $d_1$ . A circle described with centre  $F_1$  and radius  $Fd_1$  will cut the arc  $dD$  in  $D$ , a point on the curve.

Evidently by symmetry  $D_1$ , the intersection of arcs of the same radii as the above but struck from the opposite foci as centres, will also be on the curve, and so also will be the intersections on the other side of  $AA_1$ . Similarly any number of points may be found.

An alternative method may be adopted as soon as two points such as  $A$  and  $D$ , not very far apart, and the two corresponding points  $A_1$  and  $D_1$  are found. If two series of terms in geometrical progression are found,  $FA$  and  $FD$  being successive terms of the one and  $FA_1$  and  $FD_1$  successive terms of the other (Problem 8), circles struck with the corresponding terms of each as radii and with the opposite foci as centres intersect in points of the curve, the radii increasing from the one focus and diminishing from the other. This is shewn in the figure, and this construction moreover enables at once any number of ovals to be drawn, the intersection of any two circles of opposite series being taken as a starting point, and the successive intersections giving succeeding points. The second curve drawn in the figure is an example of this.

It may be noticed that a circular arc with centre at the focus coincides very closely with the oval at the vertices  $A$  and  $A_1$ .

*To draw the tangent at any point  $P$ .*

The angle  $FPG$  which the normal at any point  $P$  makes with the focal chord  $FP$  is equal to the angle which the other focal

chord  $F_1P$  makes with the chord  $CP$  drawn from  $P$  to the centre,

*The Cissoid of Diocles.*

This curve, named after Diocles, a Greek mathematician, who is supposed to have lived about the sixth century of our era, was invented by him for the purpose of constructing the solution of the problem of finding two mean proportionals. The curve is generated in the following manner:—

In the diameter  $ACB$  of the circle  $ADBE$  (fig. 157) make  $AN = BM$ , and draw  $MQ$  and  $NR$  perpendicular to  $AB$ , and let  $MQ$  meet the circle in  $Q$ , then  $AQ$  and  $NR$  intersect in a point on the curve, i.e. the locus of this intersection is the Cissoid.

By similar triangles  $\frac{RN}{AN} = \frac{QM}{AM} = \frac{\sqrt{AM \cdot MB}}{AM}$ , since  $AQB$  is a right angle; or if we call  $RN = y$ ,  $AN = x$ , and the radius of the circle  $a$ ,

$$\frac{y}{x} = \frac{\sqrt{(2a-x)x}}{2a-x} = \sqrt{\frac{x}{2a-x}},$$

which is the equation to the curve referred to rectangular axes with  $A$  as origin and  $AB$  as axis of  $x$ .

PROBLEM 155. *To describe the Cissoid corresponding to a circle of given diameter* (Fig. 157).

Of course the above description is really a construction for the curve, since by it any number of points can be determined. The curve may also be described by continuous motion thus :

Draw a diameter  $AB$  of the circle, and the tangent at  $B$ . If  $A$  is a point on the curve, this tangent will be an asymptote. Through  $C$ , the centre of the circle, draw a parallel to the tangent at  $B$  of indefinite length, and make  $AO$  on  $CA$  produced equal to  $AC$ . Cut a piece of paper to a right angle as  $abc$ , and on one side of it mark off from the angle the points  $d, c$ , making  $bd = dc = AC$ , the radius of the given circle. If the paper be now placed so that the edge  $ba$  passes through  $O$ , and the point  $c$  is always on  $ECD$ , the point  $d$  will be on the curve, and by moving it





Make  $CS$  on the diameter  $DCE = b$ . By hypothesis  $S$  will always fall beyond  $E$ . Draw  $BS$  cutting the cissoid in  $K$ . Then  $AK$  will cut  $CS$  in a point  $T$  at a distance from  $C$  equal to the required quantity  $m$ , i.e.  $CT^3 = m^3 = a^3 b$ . For draw the ordinate  $Kn$ . By similar triangles

$$\frac{CT}{Kn} = \frac{AC}{An}, \text{ or } CT^3 = \frac{Kn^3}{An^3} a^3,$$

and

$$\frac{CS}{Kn} = \frac{BC}{Bn} = \frac{BC}{2a - An},$$

$$\therefore CS = \frac{Kn}{2a - An} a;$$

but  $An$  is the  $x$  and  $Kn$  is the  $y$  of the point  $K$ , and it has been therefore already proved that

$$Kn^3 = \frac{An^3}{2a - An};$$

$$\begin{aligned} \therefore CT^3 &= \frac{An^3}{2a - An} \cdot \frac{Kn}{An^3} a^3 \\ &= \frac{Kn}{2a - An} a^3 = CSa^3 \\ &= a^3 b. \end{aligned}$$

When  $m$  is found the second mean proportional  $n$  can be found by similar triangles, for

$$a : m :: n : b.$$

If  $CS$  or  $b$  be made equal to  $2a$ ,  $m$  will be the length of the side of a cube, the volume of which is twice that of a cube of side  $a$ , since in this case  $m^3 = 2a^3$ .

*The Conchoid of Nicomedes.*

If through a fixed point  $O$  a straight line  $POp$  be drawn meeting a fixed right line  $LM$  in  $R$ , and  $RP$ ,  $Rp$  be taken each of the same constant length, the locus of  $P$  and  $p$  is called the conchoid.

If  $OD$  be drawn perpendicular to  $LM$  meeting it in  $A$ , and  $OA = a$ ,  $RP = b$ , and  $AOR = \theta$ ,

$$OP = OR + RP = \frac{a}{\cos \theta} + b.$$

Also  $Op = OR - Rp$ , since we go in the positive direction from  $O$  to  $R$ , and in the negative from  $R$  to  $p$ ;

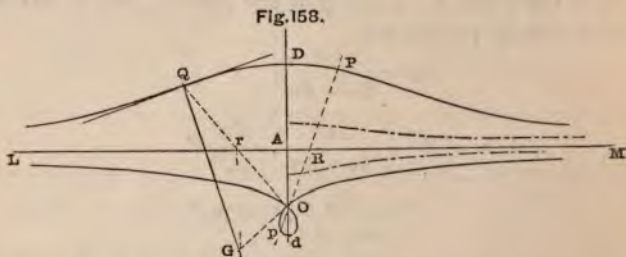
$$\therefore Op = \frac{a}{\cos \theta} - b,$$

so that the polar equation of the curve,  $O$  being the pole and  $AD$  the initial line, will be

$$(r \pm b) \cos \theta = a.$$

PROBLEM 156. *To draw the Conchoid, the constants  $a$  and  $b$  being given (Fig. 158.)*

Draw the line  $OD$ , and make  $OA$  on it  $= a$ , and  $AD$ ,  $Ad$  each  $= b$ .



Through  $A$  draw  $LAM$  perpendicular to  $OA$ ;  $LM$  will be an asymptote of the curve. Draw any line  $OP$  through  $O$  meeting  $LM$  in  $R$ , and on it make  $RP = Rp = b$ .

By definition  $P$  and  $p$  will be points on the curve, and similarly any additional number of points may be determined.

The curve is evidently symmetrical about  $OD$ .

If  $b$  is less than  $a$ , the form of the curve is that shewn by the dotted lines.

When  $b = a$  the point  $O$  is a cusp on the curve.

*To draw the normal at any point  $Q$ .*

Let  $OQ$  meet  $LM$  in  $R$ ; draw  $rG$  perpendicular to  $LM$  and  $OG$  perpendicular to  $OQ$  intersecting in  $G$ , which will be a point on the required normal; for the line  $OQ$  is moving so that it always passes through  $O$  while a fixed point on it is travelling along

$LM$ ; i.e. at the moment the line is moving along  $OQ$  (or turning about some point on  $OG$ ), and also along  $LM$  (or turning about some point in  $rg$ ), i.e.  $G$  is the centre of instantaneous rotation.

*The Witch of Agnesi.*

Let  $AB$  (fig. 159) be a diameter of a circle,  $NM$  a line perpendicular to  $AB$  meeting it in  $N$  and the circle in  $M$ . If  $P$  be taken on  $NM$  produced so that

$$\frac{PN}{AB} = \frac{MN}{AN},$$

the locus of the point  $P$  is the curve called the Witch.

If  $a$  be the radius of the circle we have from the above

$$\frac{PN^2}{4a^2} = \frac{MN^2}{AN^2} = \frac{BN}{AN} = \frac{2a - AN}{AN},$$

or putting  $AN = x$ , and  $PN = y$ ,

$$xy^2 = 4a^2(2a - x),$$

which is the equation to the curve referred to rectangular axes with  $A$  as origin and  $AB$  axis of  $x$ .

**PROBLEM 157.** *To describe the Witch of Agnesi corresponding to a circle of given diameter (Fig. 159).*

Let  $AB$  be the given diameter,  $C$  its centre; draw the tangent at  $B$ , and through  $A$  draw any number of lines  $AE, AF, \dots$  &c., cutting the circle in  $E, F$ , &c., and the tangent at  $B$  in  $e, f, \dots$  &c. Lines drawn through  $E$  and  $e$  respectively parallel and perpendicular to the tangent will intersect in  $Q$ , a point on the curve; similarly lines through  $F$  and  $f$  intersect in  $R$ , and so any number of points can be determined.

The construction is obvious from the definition of the curve.

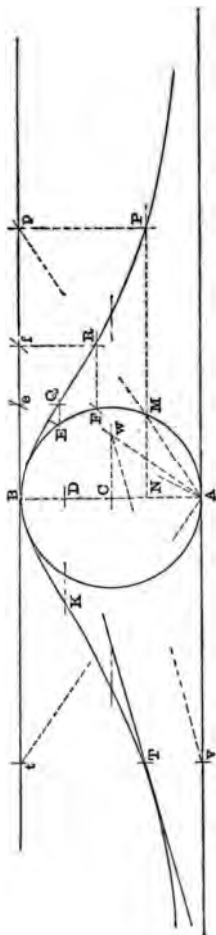
The curve is symmetrical about  $AB$  and cuts the diameter perpendicular to  $AB$  at distances from the centre equal to the diameter; the tangents at these points pass through  $B$ .

If  $CB$  be bisected in  $D$  and  $DK$  be drawn perpendicular to  $AB$  meeting the curve in  $K$ ,  $K$  is a point of inflection on the curve. The tangent to the circle at  $A$  is an asymptote to the curve.

*To draw the tangent at any point  $T$ .*

Through  $T$  draw  $tTv$  parallel to  $AB$  meeting the tangent at  $B$  in  $t$  and the asymptote in  $v$ . Draw  $Aw$  perpendicular to  $At$  meet-

Fig. 159.



ing the ordinate through  $C$ , the centre of the circle in  $w$ . The tangent at  $T$  is parallel to  $vw$ .

## THE CATENARY.

The curve in which a heavy inextensible string, freely suspended from two points, hangs under the action of gravity, is called the Catenary. If the mass of a unit length of the string is everywhere constant, i.e. if the string is of uniform density and thickness, the curve in which the string hangs is called the Common Catenary.

Investigation of the conditions of the statical equilibrium of the string gives for the curve of the common catenary the well-known equation

$$y = \frac{c}{2} \left\{ e^{\frac{x}{c}} + e^{-\frac{x}{c}} \right\},$$

the axis of  $y$  being a vertical line through the lowest point of the curve, and the axis of  $x$  a horizontal line in the plane of the string at a distance  $c$  below the lowest point.  $c$  is the length of string, the weight of which measures the tension at the lowest point, and  $e$  is the base of Napierian logarithms.

At a distance  $c$  from the origin measured along the axis of  $x$ , the corresponding value of  $y$  is

$$\frac{c}{2} \{e^1 + e^{-1}\},$$

at a distance  $2c$  it is

$$\frac{c}{2} \{e^2 + e^{-2}\},$$

and so on; and if we make  $c$  the unit of length the corresponding values of  $y$  are

$$\frac{1}{2} \{e^1 + e^{-1}\},$$

$$\frac{1}{2} \{e^2 + e^{-2}\},$$

and so on.

The third column of the following table gives the value of  $\frac{y}{c}$  at the corresponding points along the axis of  $x$  as shewn by the first column

$$e = 2.718281828... \quad \log_e 10 = .43429448...$$

| Abscissæ           | $\frac{x}{\frac{1}{2} \{e^c + e^{-\frac{x}{c}}\}}$ | $\frac{y}{c}$ |
|--------------------|----------------------------------------------------|---------------|
| $x = \frac{c}{4}$  | $\frac{1}{2} (1.28405 + .77880)$                   | 1.03142       |
| $x = \frac{c}{2}$  | $\frac{1}{2} (1.6487 + .60653)$                    | 1.1276        |
| $x = \frac{3c}{4}$ | $\frac{1}{2} (2.117 + .47144)$                     | 1.294422      |
| $x = c$            | $\frac{1}{2} (2.71828 + .36788)$                   | 1.54308       |
| $x = 2c$           | $\frac{1}{2} (7.389 + .13534)$                     | 3.76217       |
| $x = 3c$           | $\frac{1}{2} (20.0855 + .049787)$                  | 10.0676...    |
| $x = 4c$           | $\frac{1}{2} (54.598 + .018316)$                   | 27.308...     |

PROBLEM 158. *To draw the common catenary, the unit  $c$  being given.*

Example 1. ( $c = OA$ ) fig. 160.

Draw the horizontal line  $Ox$  and the vertical line  $Oy$ . On  $Oy$  measure  $OA = c$ .  $A$  will be the lowest point of the curve. Set off from  $O$  along  $Ox$  lengths  $Oa = ab = bd = c$ , and draw the ordinates through  $a, b, d...$  parallel to  $Oy$ .

On the ordinate through  $a$  measure from  $a$  a length  $ap_1 =$  (the number in third column of above table opposite  $x = c$ )  $\times c$ , i.e.  $1.54308 \times c$  (e.g. if  $c$  is  $\frac{1}{2}$ " it is only necessary to measure off on a diagonal scale of half inches a length 1.54).  $p_1$  will be a point on the curve. Similarly on the ordinate through  $b$  measure  $bp_2 =$  (number in column 3 opposite  $x = 2c$ )  $\times c$ , i.e.  $3.76217 \times c$ .  $p_2$  will be a point on the curve. Similarly for ordinate through  $d$ .

Points can of course be found between  $A$  and  $p_1$  by using the fractions of  $c$  given in the table.

Example 2. ( $c = OA$ ) fig. 161.

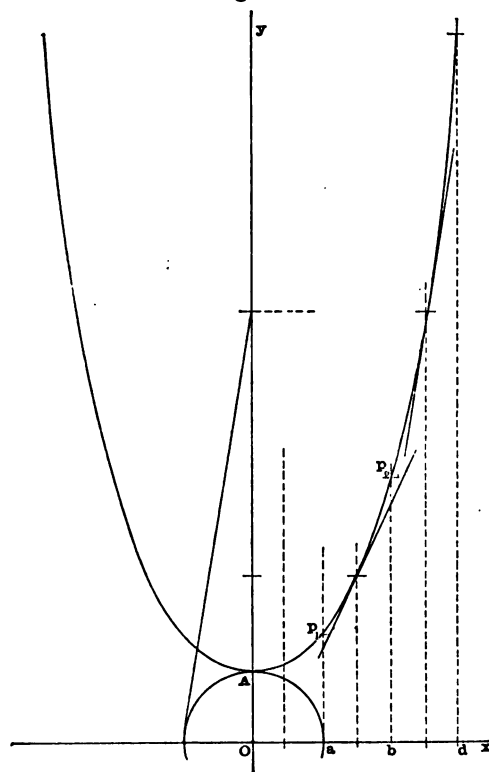
The points  $p_1, p_2, p_3, p_4$  on the ordinates through  $a, b, d, e$ , where

$$Oa = ab = bd = de = \frac{1}{4}c,$$

are given by the table: the next point furnished by the table would be on the ordinate through  $f$ , where  $ef = Oe$ . Points on

ordinates between  $e$  and  $f$  may be found without calculation as follows :

Fig. 160.



Any three equidistant ordinates  $(y_{n-1}, y_n, y_{n+1})$  are connected by the relation

$$y_{n-1} \times y_{n+1} = y_n^2 + k^2,$$

where  $k$  is some constant, i.e. if  $eg = de$

$$gp_s = \frac{\overline{ep_s}^2 + k^2}{dp_s}.$$



Construct the right-angled triangle  $AOm$ , with hypotenuse  $Am = ap_1$ , the ordinate at distance  $Oa = de = eg$  from origin: the length  $Om$  is the value of the constant  $k$ .

If  $p_4q_4$  be drawn parallel to  $Ox$  and meeting  $Oy$  in  $q_4$ ,

$$\overline{mq_4}^2 = \overline{ep_4}^2 + k^2,$$

so that the required length  $gp_5$  can be determined by taking a third proportional to  $dp_3$  and  $mq_4$ .

Similarly, if  $gh = eg = Oa$ ,

$$ep_4 : mq_5 :: mq_5 : hp_6;$$

or, since  $eh = be$ ,  $hp_6$  may be determined from

$$bp_2 : m_1q_4 :: m_1q_4 : hp_6,$$

where  $m_1$  is a point on  $Ox$  such that  $Am_1 = bp_2$ .

*To draw the tangent at any point ( $p_5$  say).*

With centre  $O$  and radius  $OA$  describe a circle; through  $p_5$  draw  $p_5q_5$  parallel to  $Ox$  and meeting  $Oy$  in  $q_5$ . The tangent at  $p_5$  will be parallel to one of the tangents which can be drawn from  $q_5$  to the above circle.

From  $g$ , the foot of the ordinate at  $p_5$ , draw  $gt$  perpendicular to the tangent at  $p_5$  meeting it in  $t$ .  $gt = OA$ , the  $c$  of the curve, and  $p_5t$  is the length of the arc of the curve between  $p_5$  and the lowest point, i.e.  $p_5t = \text{arc } Ap_5$ .

*To determine the centre and radius of curvature at any point (as  $p_5$ ).*

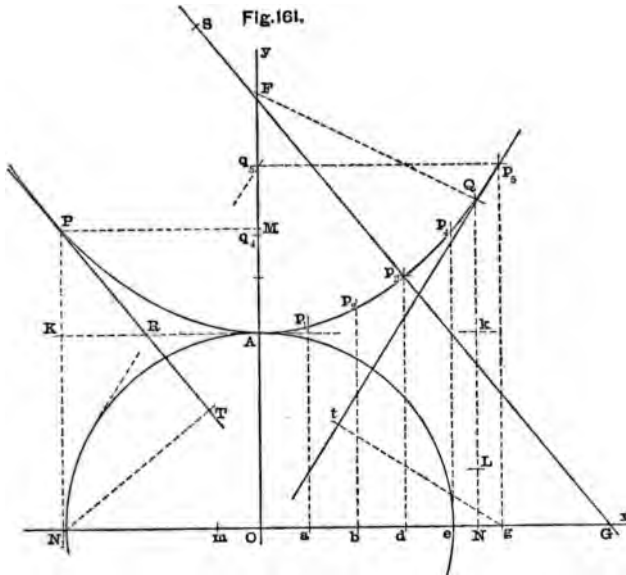
Draw the normal at  $p_5$  meeting the horizontal axis  $Ox$  in  $G$ . On the normal make  $p_5S = p_5G$ .  $S$  will be the required centre, and  $Sp_5$  the radius of curvature.

PROBLEM 159. *To draw a catenary, the vertex  $A$ , the axis  $Ay$  and a point  $Q$  being given (Fig. 161).*

The following method is approximate only, but gives tolerably close results provided the depth of  $A$  below  $Q$  does not exceed two-thirds of the distance of  $Q$  from  $Ay$ .

Find on  $Ay$  the centre ( $F$ ) of the circle passing through  $A$  and  $Q$ , and determine the length of the circular arc  $AQ$ , i.e. from a

table of the circular measure of angles get the circular measure corresponding to the number of degrees in the angle  $AFQ$  and



multiply this number by the length  $FA$  measured on any convenient scale. [In the figure  $AFQ$  contains  $64^\circ$ , the circular measure of which is  $1.117$ , and  $FA = 5$ , the unit being  $\frac{1}{4}$  inch; the length of the circular arc  $AQ$  is therefore  $5.585$  units.] From  $Q$  set off downwards on a parallel to  $Ay$  drawn through  $Q$  the length  $QL =$  the circular arc  $AQ$  as above determined, and let the horizontal through  $A$  meet  $QL$  in  $k$ , and make  $LN$  on  $Q$  produced through  $L$  a third proportional to twice  $Qk$  and  $kL$ ; i.e. take

$$LN : kL :: kL : 2.Qk.$$

$N$  will be a point on the axis  $Ox$  of the required catenary, i.e.  $c$  is determined for the required curve.

[*N* is easily determined by inflecting from *L* to *Ak* produced a length *Lk*, = twice *Qk*; produce *k*, *L* to *n* making *Ln* = *kL*.

$n$  will be a point on the required axis of  $Ox$ ; for by the similar triangles  $Lkk_1$ ,  $LNn$ ,

$$LN : Lk :: Ln : Lk_1 :: Lk : 2 \cdot Qk.]$$

The construction is based on the assumption that the length of the arc of a catenary near the vertex does not sensibly differ from the circular arc passing through its centre and extremities; and the point  $N$  is determined so that the tangent from it to a circle with centre  $Q$  and radius  $QL$  shall be equal to  $Nk$ .

PROBLEM 160. *To draw a catenary, a point of suspension  $P$ , the tangent  $PT$  at that point, and the depth  $PK$  of the loop being given (Fig. 161).*

Draw the horizontal through  $K$  meeting  $PT$  in  $R$ .

On  $PR$  produced make  $RT = RK$ , and draw  $TN_1$  perpendicular to  $PT$  meeting  $PK$  in  $N_1$ .

$KN_1$  = the unit  $c$  for the required curve.  $PT$  is the length of the arc between  $P$  and the lowest point, and a known expression for its length is

$$PT = \frac{c}{2} \left( e^{\frac{x}{c}} - e^{-\frac{x}{c}} \right),$$

where  $x = AK$ . Also

$$PN_1 = \frac{c}{2} \left( e^{\frac{x}{c}} + e^{-\frac{x}{c}} \right),$$

$$\therefore \frac{PN_1 + PT}{c} = e^{\frac{x}{c}};$$

$$\text{or} \quad \frac{x}{c} \log e = \log \overline{PN_1 + PT} - \log c,$$

$$\text{i.e.} \quad x = KN_1 \left\{ \frac{\log \overline{PN_1 + PT} - \log KN_1}{\log e} \right\},$$

which determines the vertex  $A$ .

PROBLEM 161. *To draw a catenary, the axis  $Oy$ , a point  $P$  on the curve, and the tangent  $PT$  being given (Fig. 161).*

Through  $P$  draw  $PN_1$  parallel to  $Oy$ , and  $PM$  perpendicular to  $Oy$  meeting it in  $M$ . Let the angle  $TPN_1 = \theta$ , and if  $PT$  is

= length of arc between  $P$  and  $A$  the vertex, we have if  $TN_1$  is perpendicular to  $PT$ ,

$$PT = PN_1 \cos \theta = \frac{c}{2} \left\{ \epsilon^{\frac{PM}{c}} - \epsilon^{-\frac{PM}{c}} \right\},$$

and

$$PN_1 = \frac{c}{2} \left\{ \epsilon^{\frac{PM}{c}} + \epsilon^{-\frac{PM}{c}} \right\},$$

$$\therefore PN_1 (1 + \cos \theta) = c \cdot \epsilon^{\frac{PM}{c}},$$

but

$$c = TN_1 = PN_1 \sin \theta,$$

$$\therefore \cos \frac{\theta}{2} = \sin \frac{\theta}{2} \epsilon^{\frac{PM}{c}} \quad \text{or} \quad \epsilon^{\frac{PM}{c}} = \cot \frac{\theta}{2},$$

$$\therefore \frac{PM}{c} \log \epsilon = \log \cot \frac{\theta}{2},$$

or

$$c = PM \left\{ \frac{\log \epsilon}{\log \cot \frac{\theta}{2}} \right\}.$$

By means of a table of logarithms, the value of  $c$  can be calculated, and when the length  $N_1T$  is known, the points  $N_1$  and  $T$  are of course easily determined.

#### THE TRACTORY OR ANTI-FRICTION CURVE.

The involute of the Catenary is called the Tractrix or Tractory. Since in the catenary (fig. 161)  $gt$  drawn from the foot of the ordinate at any point  $P$ , perpendicular to the tangent at  $P$ , meets it in a point  $t$  such that  $Pt$  = arc of catenary measured from the lowest point,  $t$  is evidently a point on the involute of the catenary and  $tg$  is a tangent to the involute. Also  $tg$  is constant (p. 318) and equal to  $OA$ , and therefore the Tractory is a curve such that the intercept on its tangent between the point of contact and a fixed right line is constant. This fixed length is called the *constant of the curve*.

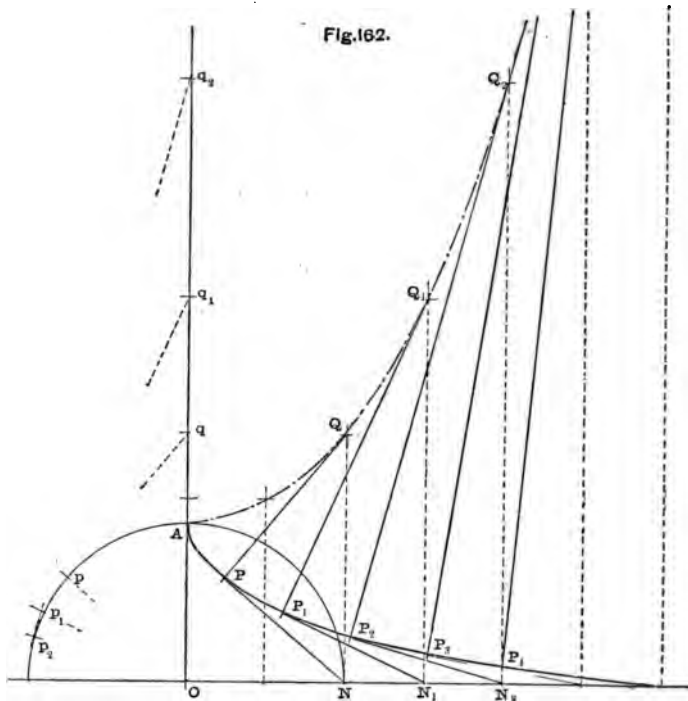
The equation of the tractory may be written

$$t \log \frac{t - \sqrt{t^2 - y^2}}{y} + x + \sqrt{t^2 - y^2} = 0,$$

where  $OA$  (fig. 162) is the axis of  $y$ ,  $ON$  the axis of  $x$  and  $OA = t$  the constant of the curve.

**PROBLEM 162.** *To draw a Tractory the constant  $t$  being given (Fig. 162).*

Describe the catenary corresponding to the unit  $t = OA$  (Problem 158).



In the figure since  $ON = OA$ ,  $QN$  the ordinate of the catenary  $= 1.543... \times OA$ , and so for other points.

Draw  $QP$  the tangent at  $Q$  (p. 318) and  $NP$  perpendicular to  $QP$  and therefore parallel to  $Op$ .  $P$  is a point on the tractrix as already shewn, and similarly other points can be determined.

The centre of curvature at  $P$  is of course the point  $Q$ .

The line  $ON$  is an asymptote to the curve, and by the revolution of the curve round  $ON$  a solid is generated, the form of which has been adopted for the foot of a vertical shaft working in a socket or step. This pivot is known as Schiele's Anti-Friction Pivot. The theoretical advantage of the adoption of the form in this case is that the vertical wear of the pivot and step is everywhere equal.

#### INVERSE CURVES.

DEF. If on any radius vector  $OP$  drawn from a fixed origin  $O$ , a point  $P'$  be taken such that the rectangle  $OP \cdot OP'$  is constant, the point  $P'$  is called the inverse of the point  $P$ ; and if  $P$  describe any curve,  $P'$  describes another curve called the inverse of the former, with respect to the pole  $O$ .

Let  $O$  be the pole and  $P, Q$  two points on any curve, and let  $P_1, Q_1$  be the inverse points, then by definition

$$OP \cdot OP_1 = OQ \cdot OQ_1 = k^2 \text{ suppose.}$$

A circle can therefore be described round  $PQQ_1P_1$  and hence the triangles  $OQP$  and  $OP_1Q_1$  are equiangular. (Euc. III. 22.)

$$\therefore \frac{PQ}{P_1Q_1} = \frac{OP}{OQ_1} = \frac{OP \cdot OQ}{OQ \cdot OQ_1} = \frac{OP \cdot OQ}{k^2}.$$

Since the angle  $OQ_1P_1$  = the angle  $OPQ$ , it follows that when  $Q$  moves up to and coincides with  $P$  so that  $PQ$  becomes the tangent at  $P$ ,  $Q_1$  moves up to and coincides with  $P_1$ , and  $Q_1P_1$  becomes the tangent at  $P_1$ , and the angle  $OP_1T_1$  between  $OP_1$  and  $Q_1P_1$  produced is equal to the angle  $OPQ$ , so that the tangents to a curve and its inverse at corresponding points make equal angles with the radius vector but on opposite sides of it.

*The Limaçon.* The inverse of an ellipse or hyperbola with respect to a focus is called a limaçon. The polar equation to an ellipse or hyperbola, the focus being the pole and the major axis the initial line, is  $r = \frac{b^2}{a} \cdot \frac{1}{1 + e \cos \theta}$ , where  $a$  and  $b$  are the major and minor axes of the ellipse or the transverse and conjugate axes of the hyperbola, and  $e$  is the eccentricity of the curve (pp.99 and 154).

If  $r$  be produced to a length  $r'$  such that  $rr' = k^2$  (Def. p. 323), the above equation becomes

$$\frac{k^2}{r'} = \frac{b^2}{a} \cdot \frac{1}{1 + e \cos \theta} \quad \text{or} \quad r' = \frac{ak^2}{b^2} (1 + e \cos \theta),$$

which is of the form  $r' = A \cos \theta + B$  the equation to the Limaçon.

$$A = \frac{ak^2}{b^2} e = \frac{ak^2}{b^2} \cdot \frac{\sqrt{a^2 \pm b^2}}{a} = \frac{k^2}{b^2} \sqrt{a^2 \pm b^2},$$

the positive sign being taken for an hyperbola, negative for an ellipse, and  $B = \frac{ak^2}{b^2}$  so that  $\frac{A}{B} = e$  the eccentricity of the conic.

Hence the constant for the Inverse being given, the values of  $A$  and  $B$  for the limaçon corresponding to any particular conic can be calculated—and conversely the equation to the Limaçon being given, and also the constant  $k$ , the particular conic of which it is the inverse may be determined by solving the above two equations for  $a$  and  $b$ .

Evidently  $A$  is less than  $B$  in the inverse of the ellipse, and greater in the inverse of the hyperbola.

PROBLEM 163. *To describe a Limaçon, the equation to the curve being given (Fig. 163).*

Let the given equation be  $r = A \cos \theta + B$ .

Draw a circle of diameter  $OD = A$ , and on  $DO$  set off from  $D$  on each side of  $D$  lengths  $DM, Dm$  each equal to  $B$ .  $M$  and  $m$  are evidently the points corresponding to the values of  $\theta$ , zero and  $180^\circ$ ,  $O$  being the pole; i.e.  $OD$  must be the initial line.







Similarly, by drawing a series of lines through  $O$  and setting off on them from the points where they cut the circle, the constant length  $B$  any number of points can be determined.

In the figure the outer curve with plain letters is the inverse of an ellipse, and the inner one with suffixed letters the inverse of an hyperbola.

The values of the constants are  $A = 2.1$ ,

$B$  for the outer curve  $= 2.4$ ,

$B$  „ inner „  $= .84$ , the unit being the length  $l$ .

Hence corresponding to the value  $k^2 = 1.7$  we have

$$2.1 = \frac{1.7}{b^2} \sqrt{a^2 - b^2},$$

and

$$2.4 = \frac{1.7}{b^2} a,$$

whence  $a = 3$  and  $b = 1.46$  the semi-axes of the ellipse of which the figure is the inverse; and corresponding to the value  $k^2 = 9$  we have

$$2.1 = \frac{9}{b^2} \sqrt{a^2 + b^2},$$

$$.84 = \frac{9a}{b^2},$$

whence

$$a = 2.03,$$

$$b = 4.66,$$

the semi-axes of the hyperbola of which the inner curve is the inverse.

*To draw the normal at any point  $P$  of a Limaçon.*

Through  $D$  draw  $DG$  parallel to  $OP$ , meeting the circle on  $OD$  as diameter again in  $G$ , which will be a point on the required normal.

*To find the centre of curvature at any point  $P$ .*

On  $OP$  as diameter describe a semicircle, and draw  $QV$  perpendicular to  $OP$  meeting it in  $V$ . On  $PG$ , the normal at  $P$ , make

$Pv = PV$  and draw  $vX$  parallel to  $GV$  meeting  $PV$  in  $X$ . On  $PV$  make  $Pg = 2 \cdot PG$  and draw  $Gx$  through  $G$  parallel to  $Pg$  and  $= PX$ .  $gx$  will intersect  $PG$  in  $s$ , the other extremity of the diameter of curvature at  $P$ , so that  $S$  the required centre is the point of bisection of  $Ps$ .

*Proof.* It is easily shewn analytically that if  $\rho$  is the radius of curvature at  $P$ ,

$$\rho = \frac{\{A^2 + 2AB \cos \theta + B^2\}^{\frac{3}{2}}}{2A^2 + 3AB \cos \theta + B^2},$$

where  $A$  and  $B$  are the constants of the curve and  $\theta$  is the angle  $DOP$ .

$$\text{But} \quad A^2 + 2AB \cos \theta + B^2 = PG^2,$$

$$\text{and} \quad \therefore A^2 + AB \cos \theta = PG^2 - (B^2 + AB \cos \theta),$$

$$\therefore \rho = \frac{PG^3}{2 \cdot PG^2 - (B^2 + AB \cos \theta)}.$$

$$\text{But} \quad QV^2 = OQ \cdot QP = AB \cos \theta,$$

$$\text{and} \quad PV^2 = PQ^2 + QV^2 = B^2 + AB \cos \theta,$$

$$\therefore \rho = \frac{PG^3}{2PG^2 - PV^2}.$$

$$\text{By construction} \quad PX : PV :: Pv : PG,$$

$$\text{and} \quad Pv = PV, \therefore PX \cdot PG = PV^2,$$

$$\therefore \rho = \frac{PG^3}{2PG - PX},$$

$$\text{i.e. } 2\rho : PG :: 2PG : 2PG - PX,$$

$$\text{or } 2\rho : 2\rho - PG :: 2PG : PX,$$

$$\text{but } Ps : Gs :: Pg : GX,$$

$$\text{i.e. } Ps : Ps - PG :: 2PG : PX,$$

$$\therefore Ps = 2\rho.$$

The limaçon is an epi-trochoid, the diameters of the directing and rolling circles being equal.



the curve, for

$$OP = OQ + QP = OD \cos DOQ + QP = A (1 + \cos \theta).$$

Similarly, any number of points on the curve can be obtained.

*To draw the normal at any point P.*

Through  $D$  draw  $DG$  parallel to  $OP$  meeting the circle again in  $G$ .  $G$  is a point on the required normal.

#### THE LEMNISCATE OF BERNOULLI.

The inverse curve of the Rectangular Hyperbola with respect to its centre is called a Lemniscate.

The polar equation to the rectangular hyperbola, the centre being the pole, and one of the axes the initial line, is

$$r^2 \cos 2\theta = a^2.$$

If any radius vector  $OP$ ,  $O$  being the centre, is produced to  $P'$  so that  $OP \cdot OP' = k^2$ , where  $k$  is any constant,  $P$  will by definition be a point on the inverse.

If  $OP = r$ ,  $OP' = r'$ , this may be written

$$r^2 r'^2 = k^4 \quad \text{or} \quad r'^2 = \frac{k^4}{r^2} \cos 2\theta;$$

the polar equation to the lemniscate may therefore be written

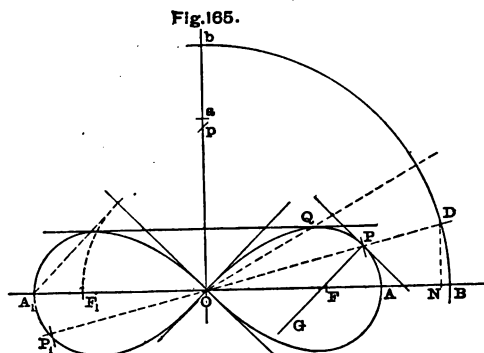
$$r^2 = K^2 \cos 2\theta.$$

The lemniscate is a particular case of the ovals of Cassini, the distance between the foci being  $\sqrt{2}K$  and the product of the focal distances of any point of the curve being  $\frac{K^2}{2}$ .

**PROBLEM 165.** *To describe a lemniscate, the constant of the curve being given (Fig. 165).*

Draw any two lines  $OB$ ,  $Ob$  at right angles to each other. On  $OB$  make  $OA = OA_1 =$  the constant  $K$  of the curve. A and  $A_1$

are evidently points on the curve corresponding to the values of



$\theta$ , zero and  $180^\circ$ . On  $Ob$  make  $Oa = OA$ , and with  $O$  as centre, and  $Aa$  as radius, describe a quadrant of a circle  $Bb$ .

Draw any line  $OD$  through  $O$  meeting the circle in  $D$ , and draw  $DN$  perpendicular to  $OA$  meeting it in  $N$ . With  $A$  as centre, and  $ON$  as radius describe an arc cutting  $Ob$  in  $p$ , and make  $OP, OP_1$  on  $OD$  each  $= Op$ .  $P$  and  $P_1$  will be points on the curve.

Similarly any additional number can be determined.

The curve passes through the origin for  $r = 0$  when  $2\theta = 90^\circ$ , and lines drawn through  $O$  making  $45^\circ$  with  $OA$  (the initial line) are tangents to the curve at  $O$ .

*Proof.* The equation to the curve may be written

$$r^2 = K^2 (2 \cos^2 \theta - 1) \text{ or } \frac{r^2 + K^2}{2K^2} = \cos^2 \theta,$$

$$\text{but } \cos^2 DON = \frac{\overline{ON}^2}{\overline{OD}^2} = \frac{\overline{ON}^2}{2K^2};$$

$$\therefore r^2 = \overline{ON}^2 - K^2,$$

which by construction it does since

$$AP = ON \text{ and } OA = K.$$

Between the values  $90^\circ$  and  $270^\circ$  for  $2\theta$ ,  $\cos 2\theta$  is negative, and consequently no real values for  $r$  exist.

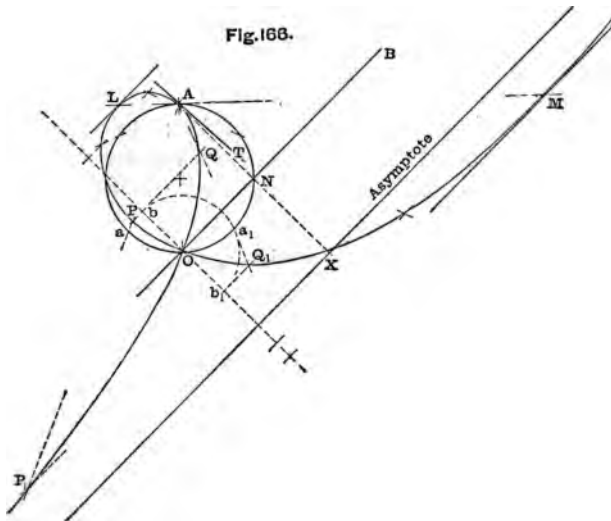
The length  $OQ$  corresponding to an angle  $AOQ = 30^\circ$  is  $\frac{1}{2}OB$ , and the tangent at  $Q$  is parallel to  $OB$ .

*To draw the tangent and normal at any point.*

The angle  $OPG$  between the radius vector  $OP$  and the normal  $PG$  is twice the angle  $POA$ . Considered as one of Cassini's ovals the foci are at  $F$  and  $F_1$ , where  $OF = OF_1 = \frac{1}{2}OB$ , and the normal may of course be drawn in the manner given for those curves, i.e. by making the angle  $F_1PG = \text{angle } OPF$ .

**PROBLEM 166.** *Given two points  $A$  and  $O$ , and a line  $OB$  through one of them, to determine the locus of a point  $P$  moving so that the angles which  $OP$  makes with  $PA$  and with a parallel to  $OB$  through  $P$ , shall be equal (Fig. 166).*

On  $OA$  as diameter describe a circle, and through  $O$  draw a perpendicular to  $OB$ .



With  $O$  as centre and any radius less than  $OA$ , describe a circle cutting the circle on  $OA$  in  $a$  and  $a_1$ , and the perpendicular through  $O$  in  $b$  and  $b_1$ .

Draw  $Aa$  meeting parallels to  $OB$  through  $b$  and  $b_1$  in  $P$  and  $P_1$ , and draw  $Aa_1$  meeting the same parallels in  $Q$  and  $Q_1$ .  $P$ ,  $Q$ ,  $P_1$  and  $Q_1$  will be points on the required locus, for the triangles  $ObQ$ ,  $Oa_1Q$ , e.g. are equal in all respects.

Similarly any additional number of points can be determined as shewn.

The curve extends to an infinite distance on both sides of  $O$ , and has an asymptote parallel to  $OB$  on the opposite side to  $A$  and at the same distance from  $OB$  as  $A$ ; or if  $AN$  be drawn perpendicular to  $OB$  and  $NX$  on it be made equal to  $AN$ , the asymptote passes through  $X$ .

The internal and external bisectors of the angle  $AOB$  are tangents at  $O$  to the two branches of the curve passing through that point. The tangent at  $A$  is inclined to  $OA$  at an angle  $OAT = \text{angle } AOB$ , and parallels to  $OB$  at distances from it  $= OA$  are tangents to the curve. The points of contact  $L$  and  $M$  of these last are determined by drawing  $LAM$  perpendicular to  $OA$ . At some point beyond  $M$  the curve becomes convex to the asymptote.

This problem is a solution of the question:—to find the point on a spherical mirror, on which a ray from any point  $A$  must impinge in order that it may be reflected parallel to a given direction.

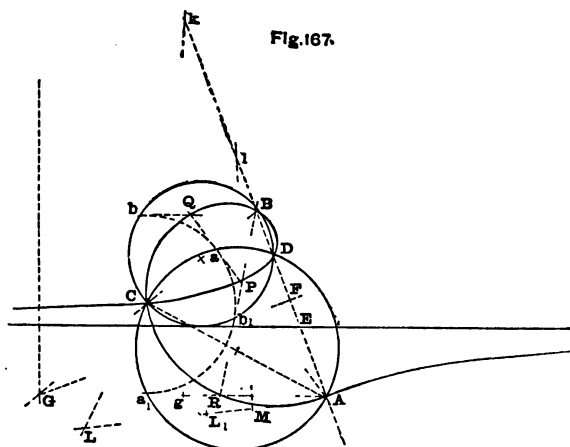
For if  $O$  be the centre of the mirror, the circular arc representing the section of the mirror by the plane passing through  $A$ ,  $O$ , and the line  $OB$  through  $O$  parallel to the given direction, will of course cut the curve in points such that the incident and reflected rays make equal angles with the normals at those points. In other words the problem is to find the point  $P$  on a given circle at which the lines  $AP$ ,  $PB$ ,  $A$  being a given point and  $PB$  being parallel to a given line make equal angles with the normal at  $P$ .

The whole curve in such a case need not be drawn, since it

is easy to find points on the curve in the neighbourhood of the part of the mirror required and to draw an arc of the curve through them.

**PROBLEM 167.** *Given three points  $A, B, C$ , to determine the locus of a point  $P$  moving so that the angles which  $PC$  makes with  $PA$  and  $PB$  are always equal (Fig. 167).*

Let  $AC$  be greater than  $BC$ . On  $AC$  and  $BC$  as diameters describe circles, and with centre  $C$  and any radius not greater than



$BC$  describe an arc cutting the circle on  $AC$  in  $a$  and  $a_1$ , and the circle on  $BC$  in  $b$  and  $b_1$ . The lines  $Aa$ ,  $Aa_1$  will intersect both the lines  $Bb$  and  $Bb_1$  in points on the required locus. Only three of the intersections are shewn in the figure, viz. the points  $P$ ,  $Q$  and  $R$ , the fourth not falling within the limits of the paper. Similarly any additional number of points can be determined as shewn.

The curve extends to an infinite distance on both sides of the line  $AB$ , and has an asymptote parallel to the line joining  $C$  to the centre point of  $AB$ , and which cuts  $AB$  between  $A$  and  $D$  the foot of the perpendicular from  $C$  on  $AB$  at a distance  $DE$  from  $D$ , which may be thus determined.



Let  $BC = a$ ,  $AC = b$ ,  $AD = m$ ,  $BD = n$  and  $CD = h$ .

It can be shewn analytically that the length

$$DE = \overline{m-n} \cdot \frac{a^2 + b^2}{\overline{m-n}^2 + 2h^2}.$$

On  $DA$  make  $DF = DB$ , therefore  $AF = \overline{m-n}$ .

Draw  $FG$  perpendicular to  $AB$  meeting  $BC$  in  $G$ , so that

$$FG = 2DC = 2h;$$

$$\therefore \overline{AG}^2 = \overline{AF}^2 + \overline{FG}^2 = \overline{m-n}^2 + 2h^2.$$

Draw  $GK$  perpendicular to  $AG$  meeting  $AB$  in  $K$ , so that by similar triangles

$$AF : AG :: AG : AK;$$

$$\therefore \overline{AG}^2 = AF \cdot AK.$$

In the figure  $K$  is beyond the limits of the paper, but if  $AG$  is bisected in  $g$ , and  $gk$  is drawn perpendicular to  $AG$  meeting  $AB$  in  $k$ ,  $AK = \frac{1}{2}AK$  and therefore  $\overline{AG}^2 = 2 \cdot AF \cdot Ak$ .

The above expression for  $DE$  therefore becomes

$$DE = \frac{a^2 + b^2}{AK}.$$

Draw  $CL$  perpendicular to  $AC$  and make  $CL = CB$  so that

$$AL^2 = a^2 + b^2.$$

On  $AB$  make  $Al = AL$ , and through  $l$  draw  $LM$  parallel to  $KL$  meeting  $AL$  in  $M$ . (In the figure  $AL$  is bisected in  $L_1$  so that  $kL_1$  is parallel to  $KL$ .) By similar triangles

$$\begin{aligned} AM : Al :: AL : AK \text{ or } AM &= \frac{Al \cdot AL}{AK} = \frac{\overline{AL}^2}{AK} \\ &= \frac{a^2 + b^2}{AK}, \end{aligned}$$

i.e.  $AM$  will be the required length  $DE$ . The asymptote can then be drawn through  $E$  parallel to the line joining  $C$  to the middle point of  $AB$ .

The internal and external bisectors of the angle  $ACB$  are tangents at  $C$  to the two branches of the curve passing through that point.

The tangents  $AT$ ,  $BT$ , at  $A$  and  $B$  make angles  $CAT$ ,  $CBT$ , with  $CA$  and  $CB$  equal respectively to the angles  $CAB$ ,  $CBA$ .

This problem is a solution of the question :—to find the point on a spherical mirror on which a ray from  $A$  must impinge in order that it may be reflected to  $B$ ;—for if  $C$  be the centre of the mirror, the circular arc representing the section of the mirror by the plane passing through  $A$ ,  $B$  and  $C$  will of course cut the curve in points such that the rays from  $A$  and  $B$  make equal angles with the normals at the points. In other words the problem is to find the point  $P$  on a given circle at which the lines  $AP$ ,  $BP$ ,  $A$  and  $B$  being given points make equal angles with the normal at  $P$ .

The whole curve in such a case need not be drawn, since it is easy to find points on the curve in the neighbourhood of the point required and to draw an arc of the curve through them.

*Magnetic curves.*

The locus of the vertex of a triangle described on a given base and having the sum of the cosines of the base angles constant, is called a magnetic curve.

If  $AB$  be the given base, and  $P$  a point on the locus, we must therefore have  $\cos PAB + \cos PBA = k$ , and corresponding to different values of  $k$ , we get a series of curves passing through  $A$  and  $B$ . These represent the *lines of force* in the plane of the paper due to a magnet whose poles are the points  $A$  and  $B$ .

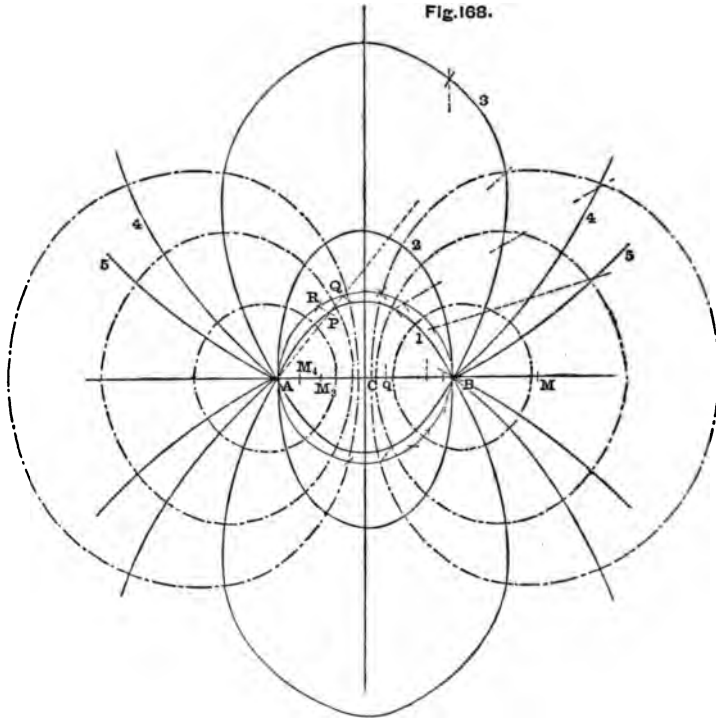
The greatest value of  $k$  is 2, since the numerical value of the cosine of an angle is never  $> 1$ , and  $k$  may have any value between 0 and 2.

PROBLEM 168. *To draw a magnetic curve, the base  $AB$  and the constant  $k$  being given (Fig. 168).*

On  $AB$  as diameter describe a circle  $ARQB$ , and on  $AB$  take a point  $M$  such that

$$AM = k \cdot AB.$$

Draw any line through  $A$  cutting the circle in  $Q$ , and make  $Aq$  on  $AM = AQ$ .



With centre  $B$ , and radius  $BR = Mq$  describe an arc cutting the circle in  $R$ .

$BR$  will intersect  $AQ$  in  $P$  a point on the required curve, for

$$\cos BAP = \frac{AQ}{AB} \quad \text{and} \quad \cos ABP = \frac{BR}{AB};$$

$$\therefore \cos BAP + \cos ABP = \frac{AQ + BR}{AB} = \frac{Aq + qM}{AB} = k.$$

Similarly, any additional number of points can be obtained.

The tangents at  $A$  and  $B$  may be determined by considering that when  $P$  moves down to  $B$  the angle  $BAP$  becomes zero, and its cosine = unity;

$$\therefore \cos ABT = k - 1.$$

In the curve marked 1 in the figure  $k = \frac{3}{2}$ ,

„ „ 2 „  $k = 1$ ,

„ „ 3 „  $k = \frac{1}{2}$ ,

„ „ 4 „  $k = \frac{1}{4}$ ,

„ „ 5 „  $k = \frac{1}{8}$ .

For curve number 2 therefore  $M$  coincides with  $B$ ,

„ 3 „  $M$  „  $C$ , the centre of the circle on  $AB$ .

For curves Nos. 4 and 5  $M$  is at  $M_2$  and  $M_4$  respectively bisecting and quadrisecting  $AC$ .

Corresponding to the value 2 of  $k$  we get the diameter  $AB$  itself for the locus, and corresponding to the value zero we get the productions of the diameter to the right and left of  $AB$ .

Each curve cuts the diameter of the circle perpendicular to  $AB$  at a distance from  $A$  or  $B = \frac{AB}{k}$ .

The chain-dotted curves in the figure are *equi-potential* curves (see next problem) and cut all the lines of force or magnetic curves at right angles.

#### *Equi-potential Curves.*

If the lines of force due to a magnet, in any plane passing through its poles, are cut normally by a series of curves, these are known as equi-potential curves, and by revolution round the line joining the poles they generate equi-potential surfaces.

If  $A$  and  $B$  are the poles of the magnet, and the length  $AB = c$ , the distances of any point  $P$  on one of the curves, from  $A$  and  $B$  are known to be connected by the relation

$$\frac{1}{AP} - \frac{1}{BP} = \frac{k}{c}$$

where  $k$  is constant throughout the particular curve considered, i.e. the equation to the series of curves may be written

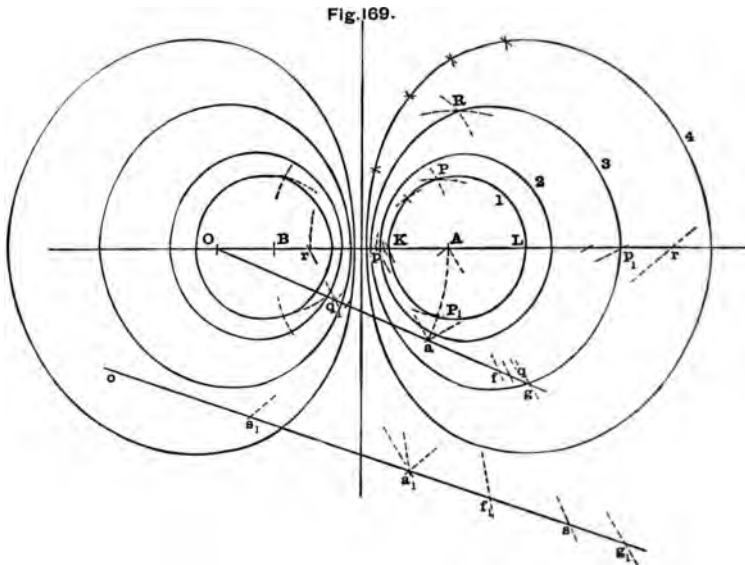
$$\frac{1}{r} - \frac{1}{r_1} = \frac{k}{c},$$

where  $r$  and  $r_1$  denote the distances of a point from  $A$  and  $B$ .

The value of  $k$  of course varies from curve to curve of the series.

**PROBLEM 169.** *To draw an equi-potential curve, the poles  $A$  and  $B$  and the constant  $k$  being given (Fig. 169).*

First determine the points in which the curve cuts the line  $AB$ .



At the point  $K$  we evidently have

$$AK + BK = c, \text{ i.e. } r + r_1 = c \text{ or } r_1 = c - r$$

which, combined with the equation

$$\frac{1}{r} - \frac{1}{r_1} = \frac{k}{c},$$

determines the value of  $r$  and  $r_1$ .

We evidently have

$$\frac{1}{r} - \frac{k}{c} = \frac{1}{r_1} = \frac{1}{c-r},$$

or

$$\frac{c-kr}{cr} = \frac{1}{c-r},$$

or

$$c^2 - (2+k)cr + kr^2 = 0$$

a quadratic to determine  $r$  or  $AK$ , but the smallest of the two roots is the only admissible solution.

At the point  $L$  we have  $BL - AL = c$ ,

$$\text{i.e. } r_1 - r = c \text{ or } r_1 = c + r.$$

The equation  $\frac{1}{r} - \frac{1}{r_1} = \frac{k}{c}$  becomes therefore in this case

$$\frac{1}{r} - \frac{k}{c} = \frac{1}{c+r} \text{ or } \frac{c-kr}{cr} = \frac{1}{c+r},$$

or

$$r^2 + cr = \frac{c^2}{k},$$

and one of the roots of this equation is the length  $AL$ .

To find any points on the curve; on  $AB$  determine a length  $AO$  such that  $AO = \frac{2c}{k}$ , i.e. take

$$AO : AB :: 2 : k.$$

Through  $O$  draw any line  $Oa$  and on it make  $Oa = OA$ ; set off on  $aO$  on each side of  $a$  equal lengths  $aq, aq_1$ ; and through  $a$  draw  $ap$  parallel to  $Aq$  meeting  $AB$  in  $p$  and also draw  $ap_1$  parallel to  $Aq_1$  meeting  $AB$  in  $p_1$ . Then  $Ap$  and  $Ap_1$  are corresponding values of  $r$  and  $r_1$  for a point on the curve and therefore a circle described with centre  $A$  and radius  $Ap$  will intersect a circle described with centre  $B$  and radius  $= Ap_1$  in points  $P$  and  $P_1$  on the curve.

The distances  $aq$ ,  $aq_1$  must be taken within certain limits, since the length  $Ap$  which depends on  $aq$  cannot be greater than  $AL$  or less than  $AK$ . These limits can evidently be determined by drawing through  $A$  a parallel to  $Ka$  meeting  $aO$  in  $f$ , and similarly drawing  $Ag$  parallel to a line through  $a$  and point  $l$  on  $AB$  such that  $Al = AL$ . The points  $q$  must then be taken between  $f$  and  $g$ .

In the figure, the value of  $k$  for the curve marked 1 is  $\frac{2}{3}$ ,

„ „ „ „ 2 „ 1,

„ „ „ „ 3 „  $\frac{1}{2}$ ,

„ „ „ „ 4 „  $\frac{1}{4}$ ,

and the corresponding values of  $AK$  and  $AL$  are

$$\text{for 1, } AK = \frac{c}{3}, \quad AL = c \cdot \frac{\sqrt{33}-3}{6},$$

$$,, 2, \quad AK = \frac{c}{2}(3-\sqrt{5}), \quad AL = \frac{c}{2}(\sqrt{5}-1),$$

$$,, 3, \quad AK = \frac{c}{2}(5-\sqrt{17}), \quad AL = c,$$

$$,, 4, \quad AK = \frac{c}{2}\{9-\sqrt{65}\}, \quad AL = \frac{c}{2}(\sqrt{17}-1).$$

These values can of course be determined arithmetically, or graphic methods may be employed.

*Proof.* From the similar triangles  $Opa$ ,  $Oaq$

$$\frac{Ap}{aq} = \frac{OA}{Oq} = \frac{l}{l+aq} \text{ if } OA=l \text{ or } \frac{1}{Ap} = \frac{l+aq}{l \cdot aq},$$

from the similar triangles  $Oap_1$ ,  $Oq_1A$

$$\frac{Ap_1}{aq_1} = \frac{OA}{Oq_1} = \frac{l}{l-aq_1} \text{ or } \frac{1}{Ap_1} = \frac{l-aq_1}{l \cdot aq_1};$$

$$\therefore \frac{1}{Ap} - \frac{1}{Ap_1} = \frac{2}{l} \text{ since } aq_1 = aq,$$

but  $l$  by construction  $= \frac{2c}{k}$ ;

$$\therefore \frac{1}{Ap} - \frac{1}{Ap_1} = \frac{k}{c}$$

or  $Ap$  and  $Ap_1$  are corresponding values of  $r$  and  $r_1$ .

It may be noticed that the line corresponding to  $Oa$  of curve 1, is, for No. 3 the line  $oa_1$ , the distance between  $A$  and the intersection of  $AB$  and  $oa_1$  being  $4 \cdot AB$ ; that the limits, between which points corresponding to  $q$  must be taken are  $f_1$  and  $g_1$ , and that the point  $R$  on the curve corresponds to  $s$  and  $s_1$  on  $oa_1$ ,  $a_1r$  being parallel to  $As$  and  $a_1r_1$  to  $As_1$ ; so that

$$AR = Ar \text{ and } BR = Ar_1.$$

The equi-potential curve corresponding to zero value of  $k$ , is the perpendicular to  $AB$  through its centre point.

#### THE CARTESIAN OVAL.

This curve owes its name to Descartes who first discussed its properties. M. Chasles, Mr Cayley, Mr Casey and others have since devoted a good deal of attention to it. A short discussion of the curve, treated geometrically, will be found in Chap. xx. of Williamson's *Differential Calculus*, 4th Edition, from which the following is mainly taken.

DEF. The locus of a point moving so that the sum or difference of its distances each multiplied by some constant from two fixed points, called the foci, is constant, is called a Cartesian oval.

If  $F, F_1$  are the two fixed points,  $P$  the moving point, and  $FP = r$ ,  $FP_1 = r_1$  and  $FF_1 = c$ , the equation of the curve may be written in either of the forms

$$nr \pm lr_1 = mc \dots\dots\dots(1),$$

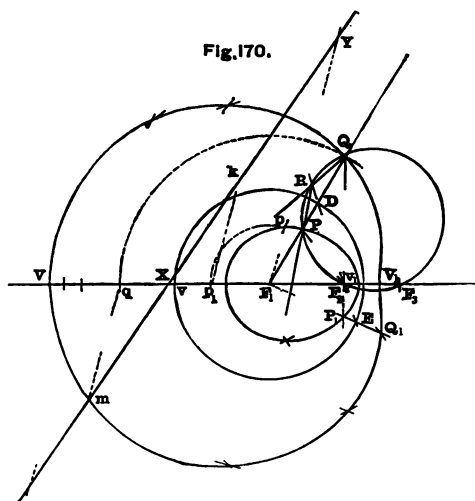
$$\text{or} \quad r \pm Mr_1 = K \dots\dots\dots(2),$$

where  $K$  is some given length and  $M$  may be assumed to be less than unity.



**PROBLEM 170.** *To draw a Cartesian oval, the foci and constants of the curve being given (Fig. 170).*

Let  $F_1$  and  $F_2$  be the given foci, and the length  $F_1F_2 = c$ . The line joining  $F_1, F_2$  is called the axis.



Let the distance of any point  $P$  on the curve from  $F_1$  be denoted by  $r_1$ , and from  $F_2$  by  $r_2$ , and suppose the equation of the curve to be written in the second of the above forms, i.e.

$$r_1 \pm Mr_2 = K.$$

On the line joining the foci, make  $F_1X = K$ , and through  $X$  draw a line  $XY$  making any convenient angle with the axis. On  $XY$  determine a length  $Xk$  such that

$$XY : F_1X :: 1 : M.$$

With centre  $F_1$  and any convenient radius less than  $F_1X$  describe an arc  $pp_1$  cutting the axis in  $p_1$ ; draw  $p_1k$  parallel to  $F_1Y$  meeting  $XY$  in  $k$ , and with centre  $F_2$  and radius  $= Xk$  describe an arc cutting the former in  $p$ ;  $p$  will be a point on the

$$r_1 + Mr_s = K;$$

$$\therefore r_s = \frac{K - r_1}{M},$$

but

$$p_1X = K - r_1,$$

and

$$Xk : p_1X :: XY : F_1X :: 1 : M,$$

i.e.

$$Xk \text{ or } r_s = \frac{K - r_1}{M}.$$

Similarly any additional number of points may be determined.

Again with centre  $F_1$  and any convenient radius greater than  $F_1X$  describe an arc  $Qq$  cutting the axis in  $q$ . Draw  $qm$  parallel to  $F_1Y$  cutting  $YX$  in  $m$ , and with centre  $F_s$  and radius  $= Xm$  describe an arc cutting the former in  $Q$ .  $Q$  will be a point on the curve, for

$$r_1 - Mr_s = K,$$

$$\therefore r_s = \frac{r_1 - K}{M};$$

but

$$Xq = r_1 - K,$$

and

$$Xm : Xq :: XY : F_1X :: 1 : M,$$

i.e.

$$Xm \text{ or } r_s = \frac{r_1 - K}{M}.$$

The curve consists of two ovals one lying wholly inside the other, the point  $p$  belonging to the inner, and  $Q$  to the outer.

The radii  $F_1p$ ,  $F_1Q$  must be taken within certain limits which may be determined thus:—

*To find the points in which the curve cuts the axis.*

Let the inner curve cut the axis in  $v$  and  $v_1$ , and the outer in  $V$  and  $V_1$ .

We have

$$r_1 \pm Mr_s = K, \quad \therefore r_s = \frac{K - r_1}{M},$$

or

$$r_s = \frac{r_1 - K}{M},$$

the positive sign referring to the inner curve, and the negative sign to the outer.

At  $v$  and  $V$  we have

$$r_1 + c_2 = r_3,$$

$$\text{i.e.} \quad F_1 v + c_2 = F_3 v = \frac{K - F_1 v}{M},$$

$$\text{or} \quad F_1 v (1 + M) = K - M \cdot c_2,$$

which determines  $F_1 v$ , and

$$F_1 V + c_2 = F_3 V = \frac{F_1 V - K}{M},$$

$$\text{or} \quad F_1 V (1 - M) = K + M \cdot c_2,$$

which determines  $F_1 V$ .

Again at  $v_1$  and  $V_1$  we have

$$r_1 + r_2 = c_2,$$

$$\text{i.e.} \quad F_1 v_1 + F_3 v_1 = c_2 = F_1 v_1 + \frac{K - F_1 v_1}{M},$$

$$\text{or} \quad F_1 v_1 (1 - M) = K - M \cdot c_2,$$

which determines  $F_1 v_1$ , and

$$F_1 V_1 + F_3 V_1 = c_2 = F_1 V_1 + \frac{F_1 V_1 - K}{M},$$

$$\text{or} \quad F_1 V_1 (1 + M) = K + M \cdot c_2,$$

which determines  $F_1 V_1$ .

The radii for points on the inner oval must be greater than  $F_1 v$  and less than  $F_1 v_1$ , and for points on the outer greater than  $F_1 V_1$  and less than  $F_1 V$ .

### *Geometrical properties of the Curve.*

The curve is evidently symmetrical about the axis.

Draw any line through  $F_1$  cutting the curve in  $P$  and  $Q$  (on the same side of  $F_1$ ); describe a circle round the triangle  $PQF_3$  cutting the axis again in  $F_2$ , then  $F_1 P \cdot F_1 Q = F_1 F_2 \cdot F_1 F_3$ ; but  $F_1 P \cdot F_1 Q$  is constant, for

$$\overline{F_3 P}^2 = \overline{F_1 P}^2 + c_2^2 - 2 F_1 P \cdot c_2 \cdot \cos F_3 F_1 P = \frac{\overline{F_1 P - K}^2}{M^2},$$

or  $\overline{F_1P}^2(1-M^2)-2(K-c_2M^2\cos F_2F_1P)F_1P-M^2\cdot c_2^2+K^2=0$ ,  
 and  $F_1P$  and  $F_1Q$  are the roots of this equation, so that their  
 product  $=\frac{K^2-M^2c_2^2}{1-M^2}$  and is constant. Hence  $F_2$  is a fixed point  
 and it possesses the same properties relative to the curve as  $F_1$  and  
 $F_3$ ; in other words  $F_2$  is a third focus. This may most con-  
 veniently be shewn from the equation of the curve in the form

$$nr_1 \pm lr_2 = mc_2,$$

where  $r_1$  is the distance of any point on it from  $F_1$ ,  $r_2$  its distance  
 from  $F_2$  and  $F_1F_2=c_2$ , and  $n>m>l$ . Let  $F_1F_2=c_2$  and denote  
 the distance of a point from  $F_2$  by  $r_2$ .

It is easily seen that the triangles  $F_1PF_2$  and  $F_1F_2Q$  are  
 equiangular;

$$\therefore \frac{F_1Q}{F_1F_2} = \frac{F_1F_2}{F_1P} \text{ and } \frac{F_2Q}{F_1F_2} = \frac{F_2P}{F_1P};$$

$\therefore$  the equation  $nF_1Q - l \cdot F_2Q = m \cdot F_1F_2$   
 may be written

$$n \cdot F_1F_2 - l \cdot F_2P = m \cdot F_1P,$$

i.e.  $m \cdot r_1 + l \cdot r_2 = n \cdot c_2$  .....(3),

which shows that the distances of any point on the inner oval  
 from  $F_1$  and  $F_2$  are connected by an equation similar in form to  
 (1) and consequently  $F_2$  is a third focus of the curve.

In like manner since the triangles  $F_1QF_2$  and  $F_1F_2P$  are equi-  
 angular, the equation

$$n \cdot F_1P + l \cdot F_2P = mF_1F_2$$

gives  $n \cdot F_1F_2 + l \cdot F_2Q = m \cdot F_1Q$ ,

or  $mr_1 - l \cdot r_2 = n \cdot c_2$  .....(4),

or the same holds for the outer oval.

Combined with the previous result, this shews that the con-  
 jugate ovals of a Cartesian referred to the two internal foci are  
 represented by the equation

$$mr_1 \pm lr_2 = n \cdot c_2$$
 .....(5),

and referred to the two extreme foci by

$$nr_1 \pm lr_2 = mc_2.$$

Similarly it is easily seen that referred to the middle and external foci, they are represented by

$$nr_2 - mr_3 = \pm lc_1 \dots\dots\dots (6),$$

where

$$c_1 = F_2F_3.$$

Taking the equation (5) referred to the two internal foci, it may be written

$$r_1 \pm \frac{l}{m} r_2 = \frac{n}{m} c_2,$$

or  $r_1 \pm A \cdot r_2 = B$  where  $A$  and  $B$  are constants.

With centre  $F_1$  and radius  $= B$  describe a circle  $DE$ .

[Evidently comparing equations (1) and (2) we may take

$$n=1, \quad l=M, \quad m=\frac{K}{F_1F_2},$$

so that

$$B \text{ or } \frac{n}{m} c_2 = \frac{F_1F_2}{K} \cdot F_1F_2,$$

i.e.

$$B : F_1F_2 :: F_1F_2 : K.]$$

Let any line through  $F_1$  meet it in  $D$  and the curve in  $P$  and  $Q$ . Let  $DF_2$  meet the circle again in  $E$ .

Now

$$PD = B - PF_1 = A \cdot PF_2,$$

$$QD = F_1Q - B = A \cdot F_2Q;$$

$$\therefore F_2Q : F_2P :: QD : DP,$$

so that  $F_2D$  bisects the angle  $PF_2Q$ .

Produce  $PF_2$  and  $QF_2$  to intersect  $F_1E$  in  $Q_1$  and  $P_1$ . The triangles  $PF_2D$  and  $P_1F_2E$  are similar and

$$\therefore \frac{P_1E}{F_2P_1} = \frac{PD}{F_2P} = A;$$

and consequently the point  $P_1$  lies on the inner oval. So also the point  $Q_1$  lies on the outer.

Again, since  $F_2D$  bisects the angle  $PF_2Q$ ,

$$\begin{aligned} F_2P \cdot F_2Q &= PD \cdot DQ + \overline{F_2D}^2 \\ &= A^2 \cdot F_2P \cdot F_2Q + \overline{F_2D}^2, \end{aligned}$$

$$\text{or} \quad (1 - A^2) F_2P \cdot F_2Q = \overline{F_2D}^2,$$

and by similar triangles  $\frac{F_2P}{F_2P_1} = \frac{F_2D}{F_2E}$ ;

$$\therefore (1 - A^2) F_2Q \cdot F_2P_1 = F_2D \cdot F_2E,$$

i.e. the rectangle under  $F_2Q$  and  $F_2P_1$  is constant; a theorem due to M. Quetelet.

If the curve has been constructed from the two internal foci, the external focus can easily be determined, for the angle  $F_1P_1F_2$  = the angle  $F_1PF_2 = F_1F_2Q$ , i.e. the angle  $F_1P_1Q$  = the angle  $F_1F_2Q$  or a circle through  $F_1P_1Q$  passes also through  $F_2$ .

*To draw the tangent and normal at any point P.*

Let  $F_1P$  meet the circle  $DE$  (of radius as previously described) in  $D$  and let  $F_2D$  meet the circle through  $PQF_2F_3$  in  $R$ . Then  $R$  is a point on the normal at  $P$  and also on the normal at  $Q$ .

They may also easily be drawn without using the circle  $DE$ .

The equation of the curve referred to the extreme foci has been shewn to be

$$nr_1 + lr_2 = mc_2.$$

On  $PF_1$ ,  $PF_2$  measure lengths  $PL$ ,  $PM$  proportional to  $n$  and  $l$  respectively, i.e. make  $PL : PM :: n : l$ .

Bisect  $LM$  in  $G$  and  $G$  will be a point on the normal at  $P$ .

The normal at  $Q$  may be constructed in exactly the same way, one of the two lengths being measured on the corresponding focal radius produced.

Similarly lengths on  $PF_1$ ,  $PF_2$  proportional to  $m$  and  $l$  determine the normal at  $P$  from vectors drawn from the internal foci.

## ELASTIC CURVES.

In the widest sense of the term, an elastic curve is the figure assumed by the longitudinal axis of an originally straight bar under any system of bending forces. It is here restricted to the figure taken by a slender flat spring of uniform section when acted upon by a pair of equal and opposite forces.

The essential property of the curve under these conditions is that the *radius of curvature at any point is inversely proportional to the perpendicular distance of that point from the line of action of the forces*. Its equation may therefore be written

$$\rho y = a^2,$$

where  $\rho$  is the radius of curvature at any point,  $y$  the distance of that point from a fixed line in the plane of the curve and  $a$  constant.

A very close approximation to the form of the curve can be easily drawn by considering it as formed of a series of circular arcs—the appropriate radius for each being determined.

**PROBLEM 171.** *To draw an elastic curve the constant of the curve and the distance of the extreme point of the loop from the line of action of the forces being given.*

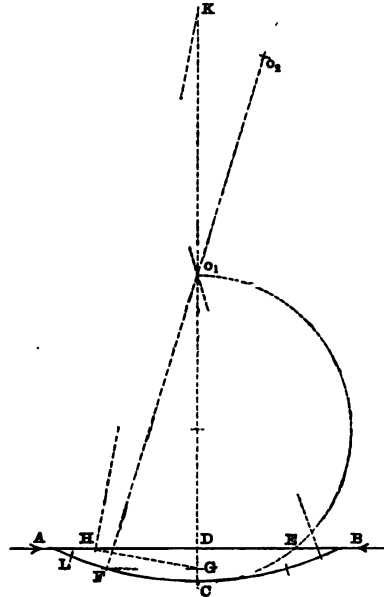
1st. A bent bow (Fig. 171).

Let  $AB$  be the line of action of the given forces,  $CD$  the maximum ordinate of the curve from  $AB$ . From any point  $D$  in  $AB$  draw  $DC$  perpendicular to  $AB$  and on it make  $DC =$  the given maximum ordinate. From  $C$  inflect to  $AB$  a length  $CE =$  the given constant of the curve and draw  $EO_1$  perpendicular to  $CE$  meeting  $CD$  in  $O_1$ . Evidently  $CD : CE :: CE : O_1C$ , so that  $O_1$  is the required centre of curvature at  $C$  and may be taken as the centre of a circular arc extending to a reasonably short distance on either side of  $C$ , draw it say to  $F$  and since  $FO_1$  is the normal at  $F$  the centre for the adjacent arc must be taken on  $FO_1$ . Draw  $FG$  parallel to  $AB$  meeting  $CD$  in  $G$  and on  $DA$  make  $DH = CE =$  the given constant of the curve.  $HK$  perpendicular to  $GH$

meets  $CO_1$  in a point  $K$  such that

$$GD : DH :: DH : DK;$$

Fig. 171.



i.e.  $DK$  is the required radius of curvature at  $F$ , and therefore if  $FO_2$  on  $FO_1$  be made equal to  $DK$ ,  $O_2$  may be taken as the centre of a circular arc extending to a reasonably short distance from  $F$  as to  $L$ . Any number of successive centres may similarly be determined.

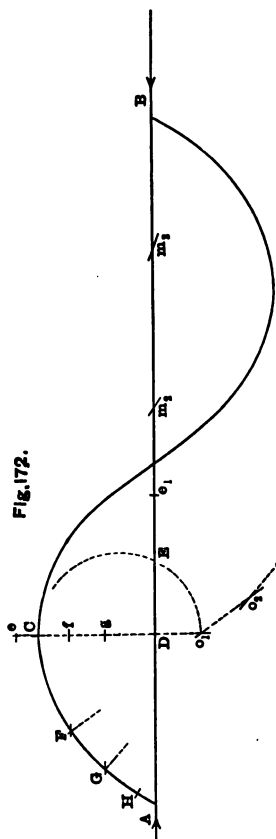
2nd. An undulating figure crossing  $AB$  at any number of intermediate points.

a. Let the given constant of the curve be greater than the maximum ordinate (Fig. 172).

Divide the given length  $AB$  into a number of equal parts corresponding to the number of required undulations and at the



centre of one such segment of the line draw  $CD$  perpendicular to  $AB$  and equal to the given maximum ordinate, from  $C$  inflect to



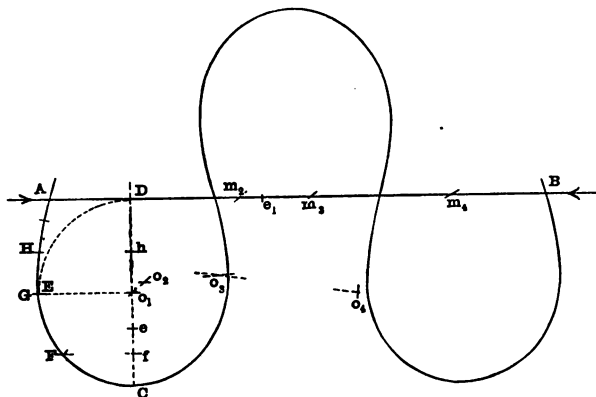
$AB$  a length  $CE$  equal to the given constant and draw  $EO_1$  perpendicular to  $CE$  meeting  $CD$  in  $O_1$ .  $O_1$  will be the required centre of curvature at  $C$  for evidently  $CD : CE :: CE : CO_1$ ; and a circular arc may be drawn through  $C$  with centre  $O_1$  and extending to a reasonably short distance on either side of  $C$  as to  $F$ . The centre of the adjacent arc must lie on  $FO_1$ . Draw  $Ff$

parallel to  $AB$  meeting  $CD$  in  $f$  and on  $DC$ ,  $DB$  respectively make  $De = De_1 = CE$ . Through  $e$  draw  $em_2$  parallel to  $fe_1$ , meeting  $AB$  in  $m_2$  and  $Dm_2$  will be the required radius of curvature at  $F$  for evidently  $Df : De_1 :: De : Dm_2$ , i.e.  $py = a^2$  where  $y$  is the ordinate of  $F$ . On  $FO_1$  make  $FO_2 = Dm_2$  and  $O_2$  may be taken as the centre of the arc adjacent to  $CF$ . Similarly any number of additional centres may be determined—supposing the second arc extends to  $G$ , draw  $Gg$  parallel to  $AB$ ,  $em_3$  parallel to  $ge_1$  and on  $GO_2$  make  $GO_3$  equal to  $Dm_3$ ,  $O_3$  will be the centre of curvature at  $G$ . As the radius of curvature at  $A$  is infinite the portion  $AH$  may be drawn tangential to the adjacent arc.

$\beta$ . Let the given constant be less than the maximum ordinate (Fig. 173).

Divide up  $AB$  and draw  $CD$  the maximum ordinate as before. On  $CD$  describe a semicircle and in it make  $CE$  equal to the

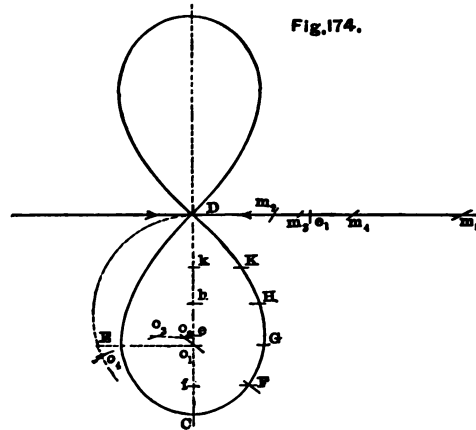
Fig. 173.



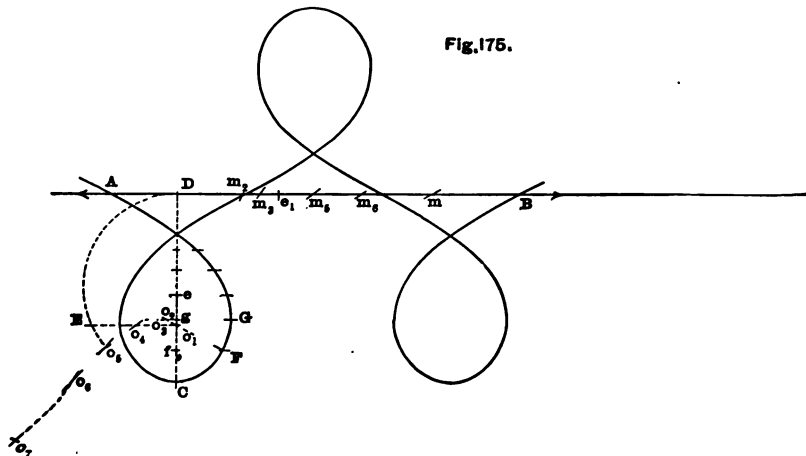
given constant: draw  $EO_1$  parallel to  $AB$  meeting  $CD$  in  $O_1$  the required centre of curvature at  $C$ . The rest of the construction is exactly similar to the above.  $De = De_1 = CE$ .  $Ff$  is parallel to  $AB$  and  $em_2$  parallel to  $fe_1$  determines  $Dm_2$  the radius at  $F$ . In the figure  $G$  is taken on  $EO_1$  so that  $g$  coincides with  $O_1$  and  $em_3$  parallel to  $O_1e_1$  determines  $Dm_3$  the radius of curvature at  $G$ .

3rd. The points  $A$  and  $B$  coinciding, which may give, with an endless spring, a figure of 8 (Fig. 174).

On  $CD$  describe a semi-circle; in it make  $CD$  equal to the given constant and draw  $EO_1$  perpendicular to  $CD$  meeting it in

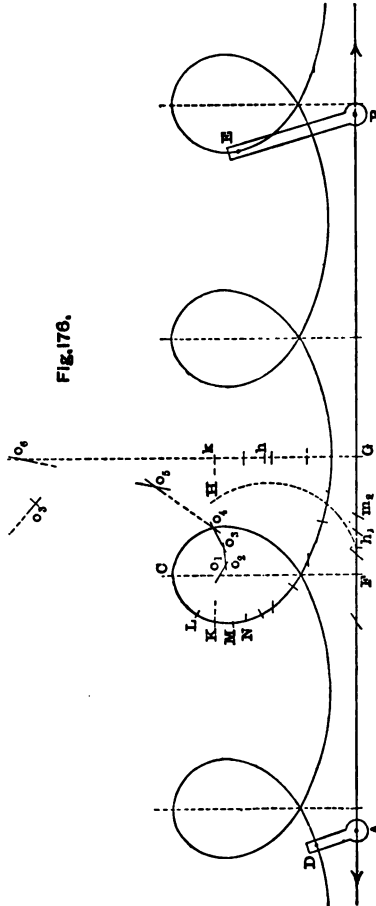


$O_1$  which will be the required centre of curvature at  $C$ . Make  $De = De_1 = CE$ ,  $De_1$  being perpendicular to  $DC$ , and successive



centres may be determined precisely as before, the curve at  $D$  being drawn tangential to the adjacent arc.

4th. In figs. 171 to 174 inclusive the forces are directed *towards* each other. When they act in directions *from* each other the spring may form one or more loops, with the ends and intermediate portions meeting or crossing  $AB$ , as shewn in fig. 175,



the construction for which is exactly similar to the preceding and which is lettered to correspond.

5th. If the forces are directed from each other at the points  $A, B$ , in two rigid levers  $AD, BE$  to which the spring is fixed at  $D$  and  $E$ , the spring forms one or more looped coils lying altogether at one side of the line of action  $AB$  (fig. 176).

The general method of construction is the same as before, but the radius of each arc corresponding to its central portion instead of to one extremity has been determined.

Let  $CF$  be a maximum ordinate; on it describe a semi-circle and in the semi-circle make  $CH$  equal to the given constant: draw  $HO_1$  perpendicular to  $CF$  meeting it in  $O_1$  the centre of curvature at  $C$ . Draw  $GO_6$  parallel to  $CF$  and at a distance from it equal to one-half the desired length of the loop of the curve, and on it make  $Gh = CH$  the given constant: make  $Gh_1$  on  $AB$  equal to  $Gh$ . Take any convenient point  $K$  at about the centre point of the intended second arc of the curve and draw  $Kk$  parallel to  $AB$  meeting  $GO_6$  in  $k$ , then  $hm_2$  drawn through  $h$  parallel to  $kh_1$  determines  $Gm_2$  the required radius of curvature at  $K$ . Take any convenient point  $L$  on the arc struck through  $C$  and join it to the centre  $O_1$ ; make  $LO_2$  on  $LO_1 = Gm_2$ , and  $O_2$  will be the required second centre. Similarly any additional number of centres can be determined.

#### CURVES OF PURSUIT.

When a point  $A$  moves so that it is continually directed towards a second point  $B$  also in motion in some known curve, the locus of  $A$  is called a "*curve of pursuit*."

The problem was first presented in the form—To find the path described by a dog which runs to overtake its master.

The velocities of the two moving points must of course be known, and the required locus can then be easily traced to any required degree of approximation by supposing the direction of

motion to be constant for a short interval and then to be suddenly deflected.

**PROBLEM 172.** *A moves in a straight line from A to B with constant velocity, and C starts from C with constant velocity double that of A and is constantly directed on A. To find the curve of pursuit (Fig. 177).*

Set off from A along AB any convenient equal distances  $A1$ ,

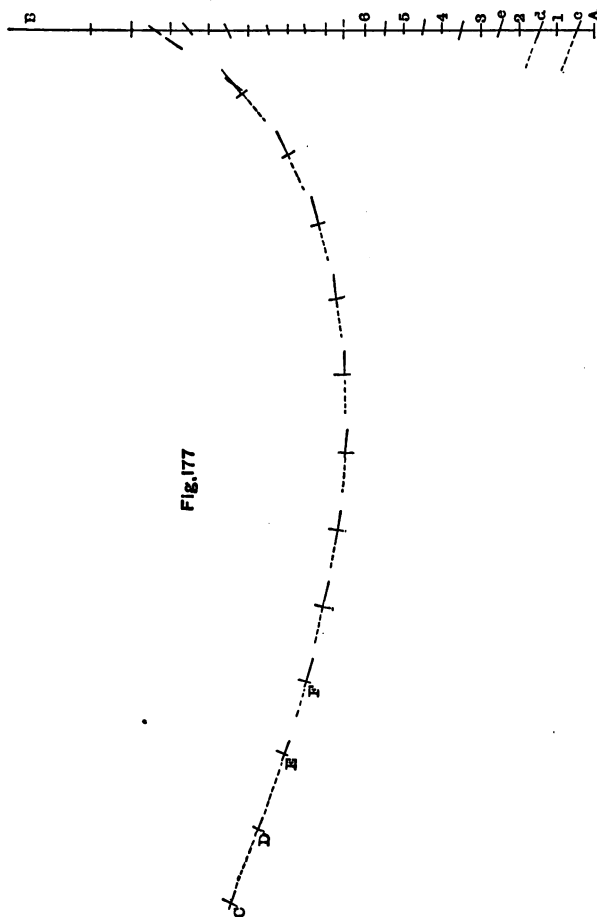


Fig. 177

12, 23,... While  $A$  advances from  $A$  to 1 suppose  $C$ 's motion to be directed on  $c$  the centre point of  $A1$ . Then when  $A$  arrives at 1,  $C$  will be at the point  $D$  on  $Cc$  such that  $CD = \text{twice } A1$ —while  $A$  advances from 1 to 2 suppose  $C$ 's motion to be directed on  $d$ , the centre point of 12; then when  $A$  is at 2  $C$  will be at the point  $E$  on  $Dd$  such that  $DE = \text{twice } 12$ , and similarly any number of successive points can be determined.

## EXAMPLES.

1. Draw a Harmonic Curve given the length  $AB$  of a vibration and a point  $P$  on the curve.

[From  $P$  draw  $PN$  perpendicular to  $AB$  meeting it in  $N$ . If  $a$  is the amplitude of the vibration  $PN = a \sin \theta$ ,

and  $\theta : 2\pi :: AN : AB$ ,

$$\therefore a = \frac{PN}{\sin \frac{2\pi \cdot AN}{AB}},$$

which determines  $a$ .

The expression  $2\pi \cdot \frac{AN}{AB}$  is the circular measure of the angle, the sine of which can then be obtained from a trigonometrical table.]

As a numerical example take  $AB = 10.8$ ,  $AN = 1.75$ ,  $PN = 1.67$ .  $a$  then equals 1.96 very approximately.

2. Draw a Cassini's oval, the foci  $F, F_1$ , and a point  $P$  on the curve being given.

[Take a mean proportional ( $k$ ) between the focal distances  $FP, F_1P$ .  $k$  is the constant of the curve. Probl. 154.]

3. Draw a Cassini's oval, the foci  $F, F_1$  and a tangent  $PT$  being given.

[Bisect  $FF_1$  in  $C$  and draw  $CT$  perpendicular to  $PT$  meeting it in  $T$ . From one of the foci  $F$  draw a line meeting  $CT$  in  $Q$

and on  $CF_1$  describe a segment of a circle containing an angle equal to the angle  $CQF$  (Prob. 30) and cutting  $FQ$  in  $p$ . The locus of  $p$  will intersect the given tangent in its point of contact, and the question reduces to the preceding. The line  $FQ$  must be drawn within certain limiting positions in order that the circle may meet it in real points.]

4. Draw through a focus  $F$  of a lemniscate a line which shall cut the curve at a given angle  $\alpha$ .

[Let  $C$  be the centre and  $F_1$  the second focus. On  $CF_1$  describe a segment of a circle containing an angle  $\frac{\pi}{2} - \alpha$ , and meeting the curve in  $P$ .  $FP$  will be the required line.]

5. Given the centre  $C$ , direction of axis  $CA$ , and a point  $P$ , on a lemniscate, draw the tangent at the point.

[Draw  $CB$  perpendicular to  $CA$ , and  $CT$  (between  $CB$  and  $CP$ ) making the angle  $BCT = \text{angle } ACP$ . Bisect  $CP$  in  $D$  and draw  $DT$  perpendicular to  $CP$ .  $T$  will be a point on the tangent at  $P$ .]

6. Describe a lemniscate with given centre  $C$ , given direction of axis  $CA$ , and to cut a given right line at a given angle.

[The direction of a tangent is obviously given. Through  $C$  draw a line parallel to this given direction, and the angle between this line and  $CB$ , perpendicular to  $CA$ , is three times the angle  $ACP$ , where  $P$  is the point in which the required tangent meets the given line.]

7. Describe a lemniscate, with given centre  $C$ , given direction of axis  $CA$ , and to pass through a given point  $P$ .

[Draw the tangent and normal at  $P$ . Ex. 5. Let the normal meet  $CA$  in  $G$ . Bisect the angle  $CPG$  by  $PD$  meeting  $CA$  in  $D$ . Through  $P$  draw lines making equal angles with  $PD$  and cutting off equal distances  $CF, CF_1$  on  $CA$ . (Prob. 19.)  $F$  and  $F_1$  are the foci of the required curve.]



8.  $ab, a'ab'$  are two lines at right angles to each other and  $a'a = ab' = \frac{1}{2}ab$ .  $ab$  moves round in the plane of the two lines till  $b$  comes to  $b'$  and  $a$  to  $a'$ , the centre point  $c$  of  $ab$  moving always along  $ca$  and a certain point  $d$  of  $ab$  describing a circular arc round  $b'$ . Determine the position of  $d$  and draw the loci of  $b$  and  $a$  throughout the motion.

9. A pendulum 5" long vibrates uniformly in an arc of  $40^\circ$ . A fly starting at the bottom crawls at a uniform speed to the top, arriving there in the time taken by a forward and backward swing of the pendulum. Trace the course of the fly.

10. A train is running in a straight line at 10 miles an hour. The door (30" wide) of one of the carriages is opened with uniform angular velocity till it stands at right angles to the direction of motion in  $\frac{1}{2}$  a second and closed again in the same time. Draw the curve traced out by a point on the edge of the door. Scale,  $\frac{1}{2} = 1$  foot (Harmonic Curve).

11.  $BD$  is a line  $1\frac{3}{4}$ " long. Draw  $AB, DC$  perpendicular to  $BD$  and each 2" long, the points  $A$  and  $C$  being on opposite sides of  $BD$ . Consider these lines as three bars jointed at  $B$  and  $D$ , and free to turn in the plane of the paper about the points  $A$  and  $C$  as centres. Trace the locus of the centre point of  $BD$ .

[The complete locus is a figure of 8, the central portion being very nearly straight lines.]

12.  $C$  is the centre of a circle,  $A$  and  $B$  are points outside the circle and in its plane. A double string is wrapped round the circle and the free loop is led off so that one portion passes round  $A$  and the other round  $B$ . Shew that any fixed point on the loop describes an hyperbola as the string is unwound by the rotation of the circle.

As a particular example take  $CA = 2''$ ,  $CB = 2''$ , diameter of circle  $\frac{7}{8}''$ , and one position of the tracing point  $1\frac{1}{4}''$  from  $A$  and  $2\frac{1}{4}''$  from  $B$ .

## CHAPTER XIII.

### SOLUTION OF EQUATIONS.

GRAPHIC methods may be applied to the solution of algebraical and trigonometrical equations, and in certain cases the process is much simpler and more expeditious than the arithmetical or analytical one. This is particularly the case with certain statical questions in which a position of equilibrium is defined by two angles for which two equations are given. "The equation for either variable which results from eliminating the other may be one of high degree, the approximate solution of which by the methods of the Theory of Equations would be very troublesome. In such cases it is often possible to obtain a solution sufficiently accurate for practical purposes by constructing curves corresponding to the equations and taking their points of intersection\*."

For example, to find  $\theta$  from the equation

$$c \sin (2\theta - a) = a \sin \theta \dots\dots\dots(1),$$

$c$ ,  $a$  and  $a$  being given constants.

If we trace the curves  $r = a \sin \theta$ ,

$$r = c \sin (2\theta - a),$$

then at their points of intersection the equation (1) is satisfied—the same origin and initial line being of course taken in tracing both loci.

\* Minchin's *Statics*, 3rd Edition, p. 49.

At first a rough tracing only is necessary, the object of this rough preliminary tracing being merely to find the places in the neighbourhood of which the curves really intersect. Then devote very special care to the tracing of the curves in these indicated neighbourhoods and in these alone. We shall thus get a value or values of the unknown variable accurate within certain narrow limits of error due to the draughtsmanship and possibility of measuring given quantities. This is as exact a solution as the graphic method pure and simple enables us to obtain, but by analysis a further step can be taken. We have obtained a near value (say  $\omega$ ) of  $\theta$ , which does not quite satisfy (1), but  $\omega + \delta$  does, where  $\delta$  is a small unknown quantity. If we write  $\omega + \delta$  for  $\theta$  in (1) and then,  $\delta$  being very small, put  $\cos \delta = 1$ ,  $\sin \delta = \delta$ , we have

$$c \sin (2\omega - a) + 2c \cos (2\omega - a) \times \delta = a \sin \omega + a \cos \omega \times \delta,$$

$$\therefore \delta = \frac{a \sin \omega - c \sin (2\omega - a)}{2c \cos (2\omega - a) - a \cos \omega},$$

so that  $\delta$  and therefore  $\omega + \delta$ , or a still nearer value of  $\theta$ , is known.

In general, if we have to solve  $F(\theta) = f(\theta)$ , i.e. any given function of  $\theta$  = to some other given function, we may trace the curves

$$r = F(\theta); \quad r = f(\theta),$$

and get an approximate value  $\omega$  of  $\theta$  from their points of intersection as above. Then the correction  $\delta$  is given by the equation

$$F(\omega) + \delta F'(\omega) = f(\omega) + \delta f'(\omega),$$

or

$$\delta = \frac{f(\omega) - F(\omega)}{F'(\omega) - f'(\omega)};$$

the dashes denoting the differential coefficients of the original functions.

*Example*.—Solve the equation  $2^\theta = 5 \sin \theta$ .

$r = 2^\theta$  represents an equiangular spiral, (Prob. 149),

$r = 5 \sin \theta$  represents a circle of diameter 5 units, passing through the origin and its centre on line through the origin perpendicular to the initial line.

Let  $\omega$  be the circular measure of the angle between the initial line and the radius drawn from the origin to a point of intersection of these curves, then

$$\delta = \frac{2^\omega - 5 \sin \omega}{5 \cos \omega - 2^\omega \log_e 2},$$

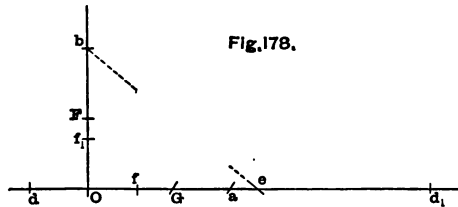
$\omega$  will be an approximate solution of the original equation; and  $\omega + \delta$  a more exact one.

**PROBLEM 173.** *To solve the quadratic equation*

$$x^2 - 2Ax + B^2 = 0 \text{ (Fig. 178).}$$

Draw two lines  $Oa$ ,  $Ob$  at right angles to each other, and on one of them make  $Ob = B$ .

With  $b$  as centre and  $A$  as radius describe an arc cutting  $Oa$  in  $a$ , so that  $Oa = \sqrt{A^2 - B^2}$ ; and with centre  $a$  and radius  $ab$



describe arcs cutting  $Oa$  in  $d$  and  $d_1$ .  $Od$  and  $Od_1$  are lines representing the two values of  $x$  in the above equation. If the numerical values of the roots are required they must be measured of course on the same scale which has been used for laying off the lengths  $A$  and  $B$ .

If  $A$  is numerically less than  $B$  the roots become imaginary, and the graphic method is not applicable.

As a numerical example we may take the equation to determine the length  $AK$  in problem 169.

Here  $AK$  is one of the roots of  $kr^2 - 2 + k[cr + c^2] = 0$ ,

$$\text{or } r^2 - \frac{2+k}{k}c \cdot r + \frac{c^2}{k} = 0,$$

which is of the above form if  $A = \frac{(2+k)c}{2k}$  and  $B = \frac{c}{\sqrt{k}}$ .

Suppose  $k = \frac{3}{2}$  then  $A = \frac{7c}{6}$  and  $B = \frac{\sqrt{2}}{\sqrt{3}}c$ ,

where  $c$  is a given length.

Make  $Oe$  in fig. 178 = this given length  $c$ .

On  $Oa$ ,  $Ob$ , take  $Of = Of_1$ , then  $ff_1$  represents  $\sqrt{2}$  the length  $Of$  being the unit; make  $OF$  on  $Ob = ff_1$ .

With centre  $f_1$  and radius = 2.  $Of$  describe an arc cutting  $Oa$  in  $G$ , then  $OG$  represents  $\sqrt{3}$ , the length  $Of$  being the unit. Through  $e$  draw a parallel to  $FG$  meeting  $Ob$  in  $b$ .

Since evidently  $Ob : Oe :: \sqrt{2} : \sqrt{3}$ ,

$Ob$  = the constant  $B$  of the equation.

With centre  $b$  and radius =  $\frac{7c}{6}$  describe an arc cutting  $Oa$  in  $a$ , and with centre  $a$  and radius  $ab$  describe arcs cutting  $Oa$  in  $d$  and  $d_1$ .  $Od$ ,  $Od_1$  represent the values of  $r$  in the equation, and the particular value of  $AK$  in Problem 169 is  $Od = \frac{Oe}{3}$ .

PROBLEM 174. *To solve the quadratic equation*

$$x^2 + 2Ax + B^2 = 0.$$

The solution is exactly the same as that of the last problem, but both roots are negative.

PROBLEM 175. *To solve the quadratic equation*

$$x^2 - 2Ax - B^2 = 0 \text{ (Fig. 179).}$$

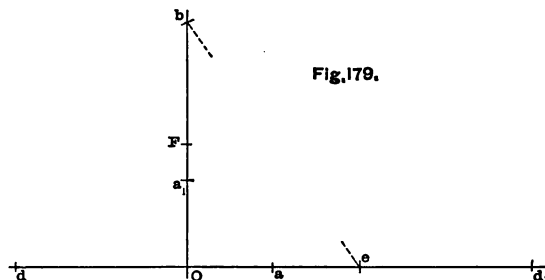
Draw 2 lines  $Oa$ ,  $Ob$  at right angles to each other and on them make  $Oa = A$ ,  $Ob = B$ :

then  $ab = \sqrt{A^2 + B^2}$ .

With centre  $a$  and radius  $ab$  describe an arc cutting  $Oa$  in  $d$

and  $d_1$ .  $Od$ ,  $Od_1$  represent the roots of the equation, but the smaller one must be taken with negative sign.

$A$  may be greater or less than  $B$ .



PROBLEM 176. *To solve the quadratic equation*

$$x^2 + 2Ax - B^2 = 0.$$

The solution is identical with that of the last problem, but the greater root must be taken with negative sign.

As a numerical example take the equation to determine the length  $AL$  in Problem 169.

$$r^2 + cr - \left(\frac{c}{\sqrt{k}}\right)^2 = 0 \text{ so that } A = \frac{c}{2} \text{ and } B = \frac{c}{\sqrt{k}},$$

suppose  $k = \frac{1}{2}$ .  $\therefore Ob : c :: \sqrt{2} : 1$ .

Make  $Oe = c$ ; bisect  $Oe$  in  $a$  and make  $Oa_1 = Oa$  so that

$$aa_1 : Oa :: \sqrt{2} : 1.$$

Make  $OF$  on  $Ob = aa_1$  and through  $e$  draw  $eb$  parallel to  $aF$ ;

then  $Ob = B$ .

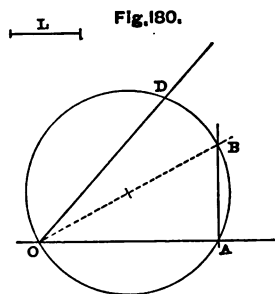
With centre  $a$  and radius  $ab$  describe arcs cutting  $Oa$  in  $d$  and  $d_1$ .  $Od$  is the positive root of the equation, and is the length  $AL$  in curve No. 3 of Problem 169.

$$a \cos \theta + b \sin \theta = c.$$

PROBLEM 177. To solve graphically the equation

$$a \cos \theta + b \sin \theta = c \text{ (Fig. 180).}$$

Draw 2 lines at right angles to each other as  $AO, AB$ . Make  $AO = a$  and  $AB = b$  on any convenient scale. Describe a circle



round  $OAB$  (its centre will of course be at the middle point of  $OB$ ) and with centre  $O$  and radius  $OD = c$  describe an arc cutting it in  $D$ , the angle  $AOD$  is the required angle  $\theta$ .

[In the figure  $a = 2.5$ ,  $b = 1.3$ ,  $c = 2.65$ , the unit being the length  $L$  and  $\theta = 47.5^\circ$ .]

$$\text{Proof.} \quad \cos BOD = \frac{OD}{OB} = \frac{c}{OB},$$

$$\cos (AOD - AOB) = \frac{c}{OB},$$

$$\cos AOD \cos AOB + \sin AOD \sin AOB = \frac{c}{OB},$$

$$\cos AOD \frac{a}{OB} + \sin AOD \frac{b}{OB} = \frac{c}{OB},$$

$$\therefore AOD = \theta.$$

The second point  $D_1$  in which the arc described with centre  $O$  and radius  $c$  would cut the circle gives when  $c$  is greater than  $a$  a second solution, the angle  $AOD_1$  being the value of  $\theta$  in this case.

When  $c$  is less than  $a$  so that  $D_1$  falls between  $O$  and  $A$  the second solution corresponds to  $a \cos \theta - b \sin \theta = c$ .





are similar branches on the other side of  $AB$  corresponding to negative values of the angles.

$$\text{Proof.} \quad \sin PAB = \frac{pB}{AB}, \text{ and } \sin PBA = \frac{Aq}{AB},$$

$$\text{i.e.} \quad \sin \theta + \sin \phi = \frac{pB + Aq}{AB} = \frac{Bb + bC}{AB} = \frac{BC}{AB} = a.$$

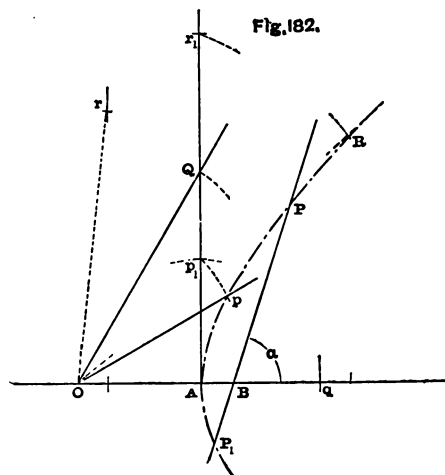
**PROBLEM 179.** To determine values of  $r$  and  $\theta$  which simultaneously satisfy the equations

$$r^2 \cos 2\theta = a^2 \dots (1), \text{ and } r \cdot \sin a - \theta = b \cdot \sin a \dots (2),$$

where  $\theta$  is the angle between the radius vector  $r$  and a fixed right line and  $a$ ,  $b$  and  $a$  are constants.

Equation (2) may be written  $\frac{r}{b} = \frac{\sin a}{\sin(a - \theta)}$ , so that  $r$  and  $b$  are evidently sides of a triangle the opposite angles of which are  $a$  (or  $\pi - a$ ) and  $a - \theta$ .

Let  $OA$  (fig. 182) be the fixed straight line from which  $\theta$  is measured,  $O$  the origin. On it make  $OB = b$  and through  $B$  draw



$BP$  making an angle  $\alpha$  with the positive direction of the initial line.  $BP$  is the locus represented by (2), for  $P$  being any point on it  $OP = r$  and  $BOP = \theta$ , so that

$$\frac{OP}{OB} = \frac{\sin OBP}{\sin BPO} = \frac{\sin \alpha}{\sin \alpha - \theta}.$$

To find points on the second locus. Make  $OA = a$ ; when  $\theta = 0$ ,  $r = \pm a$  so that the curve passes through  $A$ , and  $A_1$  on the other side of  $O$  such that  $OA_1 = a$  would be a second point on the locus. The curve is symmetrical about  $OA$  because negative values of  $\theta$  give the same  $r$  as the corresponding positive values. Through  $O$  draw any line  $Op$  and make the angle  $AOQ =$  twice the angle  $AOp$ . Draw  $AQ$  perpendicular to  $OA$  meeting  $OQ$  in  $Q$ , then  $\cos 2.AOp = \frac{OA}{OQ}$ , and  $\therefore$  if  $p$  is a point on the curve

$$Op^2 \cdot \frac{OA}{OQ} = OA^2,$$

or

$$Op^2 = OA \cdot OQ.$$

Make  $Oq$  on  $OA = OQ$  and on  $Oq$  describe a semi-circle cutting  $AQ$  in  $p_1$  and make  $Op = Op_1$ . Similarly any additional number of points on the curve may be determined, and at the points  $P$  and  $P_1$  where the line  $BP$  intersects the curve the same values of  $\theta$  and  $r$  hold for both.

As the angle  $AOp$  increases the line  $OQ$  will not intersect  $AQ$  within any reasonable distance; the length  $OQ$  may however be determined by bisecting or quadrisecting  $OA$  and taking the intersection of the ordinate through the point of division with the line corresponding to  $OQ$ —the distance of which from  $O$  will be the half or quarter of the diameter of the required semi-circle. The length  $Or$ , for example, corresponding to the radius vector  $OR$  is one-fourth the diameter of the semi-circle which determines  $r_1$  on  $AQ$  and so the length  $OR$ .

The only portions of the second locus which it is necessary to trace, are of course those in the immediate neighbourhood of

$$a \cot (\theta - \alpha) + b \cot (\phi - \beta) = c.$$

the points where it cuts the line, and a trial or two readily shews whereabouts the radii  $Op$  should be drawn.

The second locus is a rectangular hyperbola with centre  $O$  and transverse axis  $2a$ , and if this were recognised from the equation, the ordinary method of drawing an hyperbola might of course be adopted.

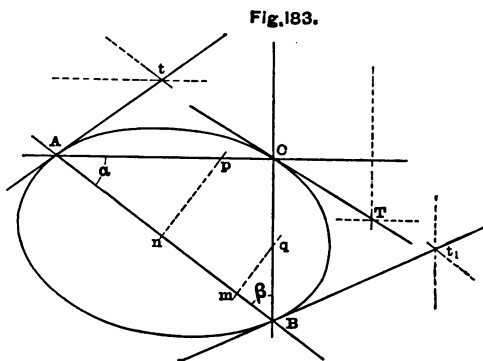
**PROBLEM 180.** *A and B are two fixed points and P a variable point, whose position is defined by the angles  $PAB (= \theta)$  and  $PBA (= \phi)$ , what locus is represented by the equation*

$$a \cot (\theta - \alpha) + b \cot (\phi - \beta) = c,$$

where  $a, b, c, \alpha, \beta$  are constants?

Equations of which the above is the general form frequently occur in statical problems, and therefore a knowledge of what it represents and how it is liable to modification may be useful (Fig. 183).

Draw  $AC, BC$  making with  $AB$  the angles  $BAC = \alpha$  and  $ABC = \beta$ . The required locus is a conic circumscribing the tri-



angle  $ABC$ , the tangents to which at those points are easily drawn.

The distance of any point  $T$  on the tangent at  $C$  from  $BC$  : its distance from  $AC$  produced  $:: b \sin \alpha : a \sin \beta$ .

If  $Ap$  on  $AC = b$  and  $pn$  be drawn perpendicular to  $AB$ ,

$$pn = b \sin \alpha,$$

and if  $Bq$  on  $BC = a$  and  $qm$  be drawn perpendicular to  $AB$ ,

$$qm = a \sin \beta.$$

$T$  can therefore be determined by drawing parallels to  $BC$  and  $AC$  at distances  $= pn$  and  $qm$  respectively.

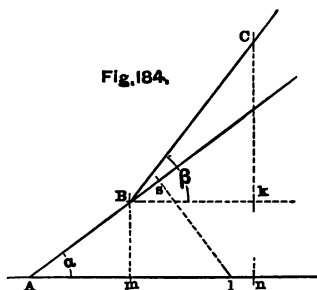
The tangent at  $A$  divides the exterior angle at  $A$  so that the distance of any point  $t$  from  $AC$  : distance from

$$AB :: a : (a \cot \alpha + b \cot \beta + c) \sin \alpha.$$

The length given by this last term is easily obtained, for if the angle  $mAB$  (fig. 184)  $= \alpha$ , and  $Bm$  perpendicular to  $Am = a$ ,  $Am = a \cot \alpha$ , draw  $Bk$  parallel to  $Am$  and make  $kBC = \beta$ , and  $Ck$  perpendicular to  $Bk = b$ ,

then

$$Bk \text{ or } mn = b \cot \beta,$$



make  $nl = c$ ,  $l$  being taken on the same side of  $n$  as  $A$  if  $c$  is negative and on the opposite side if  $c$  is positive and the length  $Al = a \cot \alpha + b \cot \beta \pm c$ : from  $l$  draw  $ls$  perpendicular to  $AB$  and  $ls = Al \sin \alpha$ .

The tangent at  $A$  is determined by drawing parallels to  $AC$ ,  $AB$  respectively at distances  $a$  and  $ls$  intersecting in  $t$ .

Similarly the tangent at  $B$  divides the exterior angle at  $B$  so that the distance of any point  $t_1$  from  $BC$  : its distance from  $AB$

$$:: b : (a \cot \alpha + b \cot \beta + c) \sin \beta.$$

The conic is therefore completely determined.



At any point  $p$  of the line we have, if  $pAq = \theta$ ,  $pBq = \phi$ ,

$$\cot \theta = \frac{Aq}{pq}, \quad \cot \phi = \frac{Bq}{pq},$$

and we want to shew therefore that

$$k \cdot Aq + l \cdot Bq = m \cdot pq \dots\dots\dots (a);$$

but 
$$\frac{pq}{CE} = \frac{Dq}{DE}, \text{ or } pq = \frac{l-k}{m} \cdot Dq,$$

$\therefore$  (a) may be written  $k(Aq + Dq) = l(Dq - Bq),$

$$\text{i.e. } k \cdot AD = l \cdot BD,$$

which by construction it does.]

To find points on the locus represented by (1). With the points  $A$  and  $B$  as centres describe two circles  $S$  and  $T$  of radii  $\frac{a}{c} \cdot AB$  and  $\frac{b}{c} \cdot AB$  respectively. Draw any common ordinate  $NLQ$ , meeting  $S$  in  $L$  and  $T$  in  $N$ ; then the lines  $AL$  and  $BN$  intersect in a point,  $P$ , on the required locus; for

$$AQ + QB = AB,$$

or  $AL \cos \theta + BN \cos \phi = AB$ , if  $BAL$  is  $\theta$  and  $ABN$  is  $\phi$ ,

or 
$$\frac{a}{c} \cdot AB \cdot \cos \theta + \frac{b}{c} \cdot AB \cos \phi = AB,$$

which is the given equation.

If the line  $DC$  meet the curve in  $R$  and  $R_1$ , the angles  $RAB$ ,  $R_1AB$  are the required values of  $\theta$ , and the angles  $RBA$ ,  $R_1BA$  those of  $\phi$ .

There is a precisely similar loop on the other side of  $AB$ .

In the particular case in which  $a = b$  the locus is the *Magnetic Curve*. (Prob. 168.)

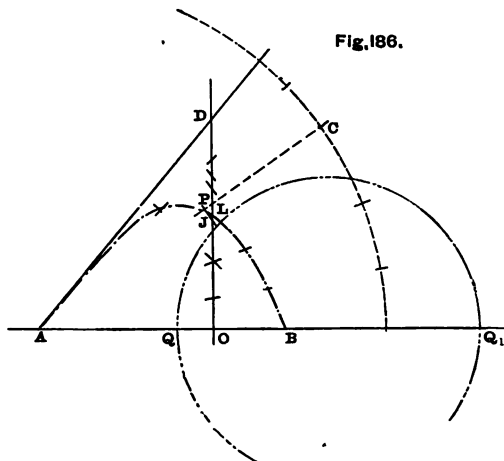
PROBLEM 182. To find  $\theta$  and  $\phi$  from the equations

$$\frac{a}{\sin \theta} + \frac{b}{\sin \phi} = c \dots (1), \text{ and } \cos \theta = k \cos \phi \dots (2),$$

where  $a$ ,  $b$ ,  $c$ ,  $k$  are constants (Fig. 186).

372 \quad \frac{a}{\sin \theta} + \frac{b}{\sin \phi} = c \text{ and } \cos \theta = k \cos \phi.

Take two points  $A$  and  $B$  such that  $AB = \overline{a+b}$ ; make  $AO = a$ ,  $OB = b$  and draw  $OD$  perpendicular to  $AB$ ; with  $A$  as centre and



$c$  as radius describe a circle, and draw any radius  $AC$  meeting  $OD$  in  $L$ ; inflect  $BJ = LC$  ( $J$  being on  $OD$ ); then  $P$ , the point of intersection of  $AC$  and  $BJ$  is a point on the locus represented by (1), the angles  $\theta$  and  $\phi$  being  $ALO$  and  $BJO$  respectively.

There is a precisely similar loop on the other side of  $AB$ .

Again the equation  $\cos \theta = k \cos \phi$  gives  $\sin PAB = k \cdot \sin PBA$  or  $PB = k \cdot PA$ , i.e.  $P$  is the vertex of a triangle on a given base  $AB$  and with sides in a given ratio (Problem 17), i.e. the locus represented by the second equation is a circle whose diameter  $QQ_1$  is the line joining the points which divide  $AB$  internally and externally in the ratio  $1 : k$ ; i.e.

$$AQ : QB :: 1 : k :: AQ_1 : Q_1B.$$

The values of  $\theta$  and  $\phi$  which satisfy both equations are those belonging to the points of intersection of this circle and the previous curve.

## EXAMPLES.

1. Solve the equation  $\frac{\sin x}{x} = \frac{1}{2}$ .

[Trace the loci  $y = \sin x$  (harmonic curve) and  $y = \frac{x}{2}$  (a straight line through the origin): the values of  $x$  corresponding to their points of intersection are solutions.]

2. Solve the equation  $\sin x = ax + b$ .

[The intersections of the harmonic curve  $y = \sin x$  and of the straight line  $y = ax + b$  where  $a$  and  $b$  are constants.]

3. Solve the equation  $2^\theta = 5 \sin \theta$ .

[The intersections of the equiangular spiral  $r = 2^\theta$  and of the circle  $r = 5 \sin \theta$ .]

4. Find  $\theta$  and  $\phi$  from the equations

$$\tan \phi = n \tan \theta \dots \dots \dots (1),$$

$$a \cos \theta = b \cos \phi + c \dots \dots \dots (2),$$

where  $a$ ,  $b$ ,  $c$  and  $n$  are given constants.

[The 2nd equation represents a locus identical with (1) of Problem 181, attention being paid to the usual conventions as to sign.

The 1st equation represents a right line perpendicular to  $AB$  (fig. 185), the base of this locus, and meeting it in  $D$  so that

$$AD = n \cdot BD.]$$

5. Find  $\theta$  and  $\phi$  from the equations

$$l \cos \theta + n \cos \phi = a - m \cos \alpha,$$

$$l \sin \theta - n \sin \phi = m \sin \alpha$$

where  $l$ ,  $m$ ,  $n$ ,  $a$  and  $\alpha$  are constants.

[Draw two lines  $AB$ ,  $AC$  including an angle  $\alpha$ , and make  $AB = a$  and  $AC = m$ . With centres  $B$  and  $C$  and radii  $= r$



respectively describe arcs intersecting in  $D$  on the same side of  $BC$  as  $AB$ . Let  $CD$  meet  $AB$  in  $E$ .  $BED$  is the required value of  $\theta$  and  $DBE$  that of  $\phi$ .]

6. Determine  $\theta$  from the equation

$$a \cos \lambda \cdot \cos (\lambda + 2\theta) = c \cdot \cos (\alpha + \theta)$$

where  $a$ ,  $c$ ,  $\lambda$  and  $\alpha$  are given constants.

[The locus represented by the right-hand side of the above equation is a circle of radius  $c$ , the origin ( $O$ ) being the extremity of a diameter, and the initial line making an angle  $\alpha$  therewith.

To draw the locus represented by the left-hand side :—draw a line through the origin  $O$  making an angle  $\lambda$  with the initial line, and on it measure a length  $OL = a$ . Draw  $LN$  perpendicular to the initial line meeting it in  $N$  so that  $ON = a \cos \lambda$ . With centre  $O$  and radius  $ON$  describe a circle. From any point  $Q$  on this circle draw  $QM$  perpendicular to  $OL$  meeting it in  $M$ . Draw  $OP$  bisecting the angle  $NOQ$  and make  $OP = OM$ .  $P$  will be a point on the second locus, and any additional number of points may be similarly determined. Let the two loci intersect in  $X$ , and the angle between  $OX$  and the initial line is the required angle  $\theta$ .]

This equation defines the position of equilibrium of a uniform rectangular board resting in a vertical plane against two equally rough pegs in a horizontal line.

THE END.

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